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تقريب تحليلي لثابت لانداو باستخدام طريقة مفكوك متعدد حدود بوبكر

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الملخص:

في هذا البحث وردت صيغ تقريبية لتقييم ثابت لانداو باستخدام طريقة مفكوك متعدد حدود بوبكر (BPES). كما تم مقارنة النتائج هذه الدراسة مع الدراسات التي تمت الاشارة اليها.

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ORIGINAL ARTICLE

Analytical approximation for Landau's constants by using the Boubaker Polynomials Expansion Scheme method

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Error analysis

Abstract In this study, approximation formulas for evaluating Landau constants are elaborated by using the Boubaker Polynomials Expansion Scheme (BPES). Results are compared to some referred studies.

1. Introduction

Landau's constants are defined (Landau, 1913; Popa, 2010; Zhao, 2009) for all positive integers n, by:

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \cdots + \left(\frac{(2n-1)!!}{(2n)!!}\right)^2. \tag{1.1}$$

Some properties of Landau's constants are obtained and studied in (Alzer, 2002; Cvijović and Srivastava, 2009; Chen, 2012; Granath, 2012; Mortici, 2011; Nemes, 2012; Popa and Secelean, 2011). These constants were defined in relation with a set F of complex analytic functions f defined on an open region containing the closure of the unit disk D ($D = z : |z| < 1$) satisfying the conditions:

$$f(z)|_{z=0} = 0 \quad \text{and} \quad \frac{d}{dz} f(z)|_{z=0} = 1. \tag{1.2}$$

If $\ell(f)$ is the supremum of all numbers such that $f(D)$ contains a disk of radius 1, and that f verifies the additional conditions:

$$|f(z) < 1(|z| < 1) \quad \text{and} \quad f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1.3}$$

then:

$$L = \inf\{\ell(f) : f \in F\} = \left| \sum_{k=0}^{\infty} a_k \right| \leqslant G_n. \tag{1.4}$$

In this paper, a convergent protocol is proposed in order to give analytical expressions to the Landau's constants.

2. Resolution process

In concordance of the approximations performed by Watson, 1930, Zhao, 2009 and Popa, 2010, a sequence $\{\omega_n\}_{n \geqslant 0}$ is defined as:

$$\omega_n = G_n - \frac{1}{\pi} \ln(n + A) - \frac{1}{\pi}(\gamma + \ln 16) + \frac{B}{n + C}, \tag{2.1}$$

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where γ is Euler's constant and A, B and C are unknown constants.

The resolution process aims to find accurate approximation to the values of the parameters A, B and C such that $\{\omega_n\}_{n\geqslant 0}$ is the fastest sequence which would converge to zero.

3. Approximation using The Boubaker Polynomials Expansion Scheme (BPES)

3.1. Presentation

The Boubaker Polynomials Expansion Scheme BPES (Milgram, 2011; Rahmanov, 2011) is a resolution protocol which has been successfully applied to several applied-physics and mathematics problems. The BPES protocol ensures the validity of the related boundary conditions regardless of the features of the main equation. The BPES is mainly based on the properties of the Boubaker polynomial's first derivatives:

$$\sum_{k=1}^{N} B_{4k}(x)\bigg|_{x=0} = -2N \neq 0, \quad \sum_{k=1}^{N} B_{4k}(x)\bigg|_{x=r_k} = 0 \tag{3.1}$$

and

$$\sum_{k=1}^{N} \frac{dB_{4k}(x)}{dx}\bigg|_{x=0} = 0, \quad \sum_{k=1}^{N} \frac{dB_{4k}(x)}{dx}\bigg|_{x=0} = \sum_{k=1}^{N} H_k \tag{3.2}$$

with

$$H_n = B_{4n}(r_n) = \frac{4\alpha_n[2 - r_n^2]\sum_{k=1}^{n} B_{4k}^2(r_n)}{B_{4(n+1)}(r_n)} + 4r_n^3,$$

where r_k are B_{4k} minimal positive roots. Several solution have been proposed through the BPES in many fields such as numerical analysis (Milgram, 2011), theoretical physics, mathematical algorithms, heat transfer, homodynamic, material characterization, fuzzy systems modeling and biology (Rahmanov, 2011).

3.2. Application

The resolution protocol is based on setting $\widehat{A}, \widehat{B}$ and $\widehat{C}$ as estimators to the constants A, B and C, respectively:

$$\begin{cases} \widehat{A} = \frac{1}{2N_0}\sum_{k=1}^{N_0} \xi_k^A B_{4k}(xr_k), \\ \widehat{B} = \frac{1}{2N_0}\sum_{k=1}^{N_0} \xi_k^B B_{4k}(xr_k), \\ \widehat{C} = \frac{1}{2N_0}\sum_{k=1}^{N_0} \xi_k^C B_{4k}(xr_k), \end{cases} \tag{3.3}$$

where B_{4k} are the $4k$-order Boubaker polynomials, N_0 is a prefixed integer, and $a_k|_{k=1,\ldots,N_0}$ are unknown pondering real coefficients. As a first step, the coefficients, $\xi_k^A|_{k=1,\ldots,N_0}$ are determined through Falaleev approximation (Falaleev, 1991):

$$G_n \approx \frac{1}{\pi}\left[\ln\left(n + \frac{3}{4}\right) + \gamma + \ln 16\right]. \tag{3.4}$$

The BPES solution for $\widehat{A}$ is obtained by determining the non-null set of coefficients $\widehat{\xi}_k^A|_{k=1,\ldots,N_0}$ that minimizes the absolute difference Δ_{N_0}:

$$\Delta_{N_0} = \left|\frac{1}{2N_0}\sum_{k=1}^{N_0} \widehat{\xi}_k^A \Lambda_k - \frac{1}{\pi}(\gamma + \ln 16)\right| \tag{3.5}$$

with

$$\Lambda_k = \frac{3}{4}\int_0^1 \sum_{k=1}^{N_0} x B_{4k}(xr_k)dx.$$

Values of $\widehat{B}$ and $\widehat{C}$ are consecutively deduced from coefficients $\widehat{\xi}_k^B|_{k=1,\ldots,N_0}$ and $\widehat{\xi}_k^C|_{k=1,\ldots,N_0}$ which minimize the absolute difference Δ'_{N_0}:

$$\Delta'_{N_0} = \left|\frac{1}{2N_0}\sum_{k=1}^{N_0} \widehat{\xi}_k^A X_k - \frac{1}{\pi}(\gamma + \ln 16) + \frac{1}{2N_0}\sum_{k=1}^{N_0} \widehat{\xi}_k^B Y_k \left(1 - \frac{1}{2N_0}\sum_{k=1}^{N_0} \widehat{\xi}_k^C Z_k\right)\right| \tag{3.6}$$

with

Table 1 Landau constant values.

n	Landau constants			Quadratic error vs. exact	
	Falaleev Approximation (Falaleev, 1991)	Brutman Approximation Brutman, 1982	BPES Approximation	Ref. Falaleev, 1991	Ref. Brutman, 1982
0	0.97463516	0.97469795	0.97473411	9.7924E-9	1.30765E-9
1	1.24433844	1.24440124	1.24449240	2.37026E-8	8.31042E-9
2	1.38820978	1.38827257	1.38841873	4.36628E-8	2.13632E-8
3	1.48693516	1.48699795	1.48719911	6.96731E-8	4.04659E-8
4	1.56218004	1.56224284	1.56249900	1.01733E-7	6.56187E-8
5	1.62299481	1.62305761	1.62336877	1.39843E-7	9.68215E-8
6	1.67403346	1.67409626	1.67446242	1.84004E-7	1.34074E-7
7	1.71800808	1.71807088	1.71849204	2.34214E-7	1.77377E-7
8	1.75663844	1.75670124	1.75717740	2.90474E-7	2.2673E-7
9	1.79108390	1.79114669	1.79167785	3.52784E-7	2.82133E-7
10	1.82216319	1.82222598	1.82281214	4.21145E-7	3.43585E-7
11	1.85047605	1.85053884	1.85118001	4.95555E-7	4.11088E-7
12	1.87647497	1.87653777	1.87723393	5.76015E-7	4.84641E-7
13	1.90050977	1.90057257	1.90132373	6.62525E-7	5.64244E-7
14	1.92285648	1.92291928	1.92372544	7.55085E-7	6.49896E-7
15	1.94373675	1.94379954	1.94466070	8.53696E-7	7.41599E-7
16	1.96333123	1.96339403	1.96431019	9.58356E-7	8.39352E-7
17	1.98178915	1.98185195	1.98282311	1.06907E-6	9.43155E-7
18	1.99923515	1.99929795	2.00032411	1.18583E-6	1.05301E-6
19	2.01577445	2.01583725	2.01691841	1.30864E-6	1.16891E-6

$$X_k = \frac{3}{4\pi}\int_0^1 \sum_{k=1}^{N_0} x B_{4k}(xr_k)dx \quad \text{and} \quad Y_k = Z_k$$
$$= \int_0^1 \sum_{k=1}^{N_0} B_{4k}(xr_k)dx.$$

Hence the final solution is:

$$G_n = \frac{1}{\pi}\ln(n + A) + \frac{1}{\pi}(\gamma + \ln 16) - \frac{B}{n + C} \tag{3.7}$$

with

$$\begin{cases} A = \widehat{A} = \frac{1}{2N_0}\sum_{k=1}^{N_0} \xi_k^A B_{4k}(xr_k), \\ B = \widehat{B} = \frac{1}{2N_0}\sum_{k=1}^{N_0} \xi_k^B B_{4k}(xr_k), \\ C = \widehat{C} = \frac{1}{2N_0}\sum_{k=1}^{N_0} \xi_k^C B_{4k}(xr_k). \end{cases}$$

4. Results, plots and discussion

Numerical solutions obtained by the given method are gathered in Table 1 along with precedent refereed approximations by Falaleev, 1991 and Brutman, 1982.

Plots of the BPES solution are presented in Fig. 1, along with referred solutions (Falaleev, 1991; Brutman, 1982).

Fig. 2 displays the errors of the obtained values along with those recorded by Falaleev, 1991 and Brutman, 1982. For accuracy purposes, error analysis has been carried out for the two referred datasets. Examination of the quadratic error plots (Fig. 2) shows that the amplitudes of the error are more exaggerated according to Falaleev approximation (Falaleev, 1991), particularly for low values of n.

Moreover, Fig. 2 monitors an obvious logarithmic profile in the range with a quadrature error which does not exceed 3.5%

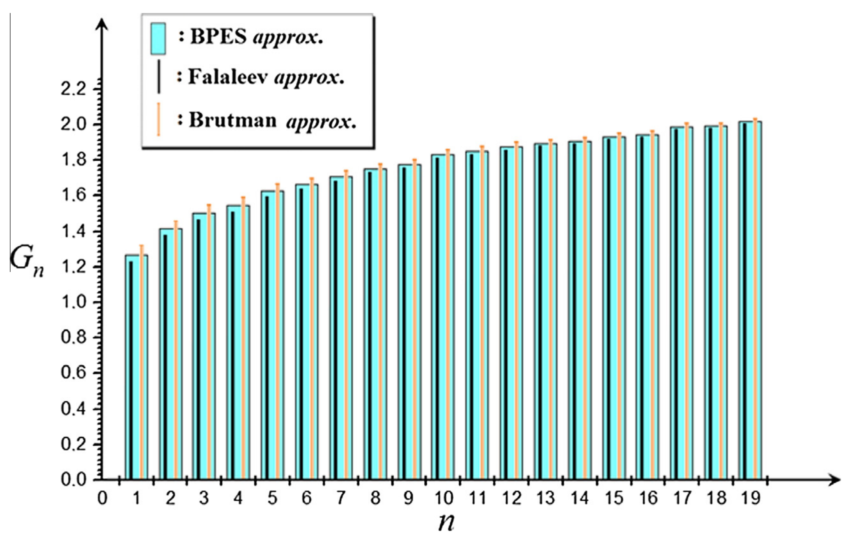

Fig. 1 Plots of the BPES solution for Landau's constants alongwith referred solutions (Falaleev, 1991,Brutman, 1982)

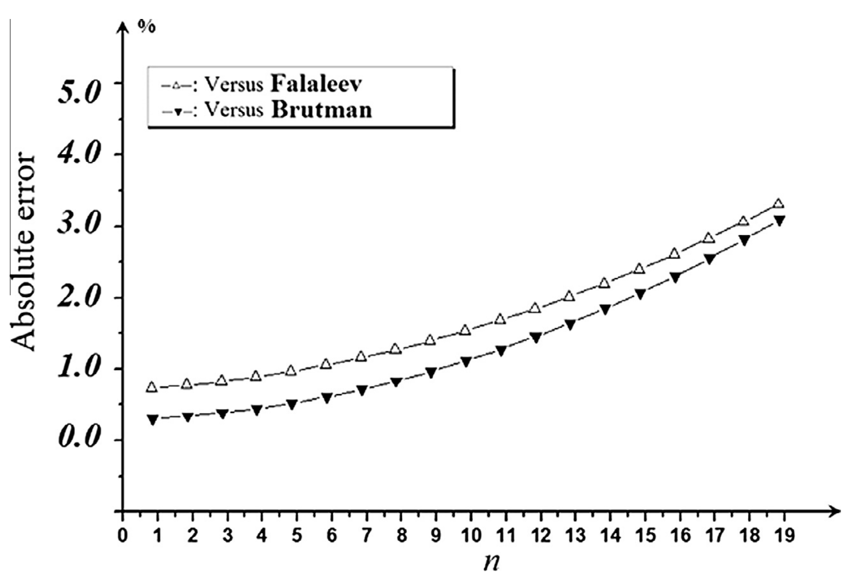

Fig. 2 Plots of solution errors versus that of Falaleev and Brutman.

on the whole range (Fig. 2). The obtained profile is in good agreement with the values recorded elsewhere (Falaleev, 1991; Brutman, 1982; Cvijovic and Klinowski, 2000).

5. Conclusion

In this study, approximation formulas for evaluating Landau constants have been presented and discussed. The proposed estimate has been compared with two other estimates which are of special importance in approximation theory. Results have been favorable for the performed method in terms of both convergence and accuracy.

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