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Minimal spaces with a mathematical structure

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Abstract This paper will discuss, grill topological space which is not only a space for obtaining a new topology but generalized grill space also gives a new topology. This has been discussed with the help of two operators in minimal spaces.

1. Introduction

The concept of grill is well known in topological spaces. Choquet (see Choquet, 1947) introduced this notion in 1947. After that mathematicians like Thrown (see Thrown, 1973) and Chattopadhyay and Thron (1977) and Chattopadhyay et al. (1983) have developed this study at closure spaces, compact spaces, proximity spaces, uniform spaces and many other spaces. Grill topological space (see Choquet, 1947) is a much more new concept in the literature. It was introduced by Roy and Mukherjee (2007) in 2007. Authors like Al-Omari and Noiri (2011, 2012b, 2013), Hatir and Jafari (2010) and Modak(2013a,b,c,d) have studied this field in detail. They have concentrated their study on two operators and generalized sets on this space and obtained different topologies. Modak has shown that new topology can be made from various types of generalized spaces in Modak (2013b,c).

In this paper we shall define two operators on Alimohammady and Roohi's minimal space (see Alimohammady and Roohi, 2005). We also divide the properties of these two operators into two parts. Again we shall try to obtain a new topology with the help of this minimal space with a grill on the same space. However Roy and Mukherjee (2007) and Al-Omari and Noiri (2012a,b) have considered grill topological space for obtaining a new topology. Actually throughout this paper, we are trying to catch the essential space with a grill on the same space that gives a new topology.

2. Preliminaries

Following are the preliminaries for this paper:

Definition 2.1 (Alimohammady and Roohi, 2005). A family $\mathcal{M} \subseteq \wp(X)$ is said to be minimal structure on X if $\emptyset, X \in \mathcal{M}$.

In this case $(X, \mathcal{M})$ is called a minimal space. Throughout this paper $(X, \mathcal{M})$ means minimal space.

Example 2.2 (Alimohammady and Roohi, 2005). Let (X, τ) be a topological space. Then $\mathcal{M} = \tau$, $SO(X)$ (Levine, 1963), $PO(X)$ (Mashhour et al., 1982) and $\alpha O(X)$ (Njastad, 1965) are the examples of minimal structures on X.

Recall the definition of grill:

Definition 2.3 (Choquet, 1947). A subcollection $\mathcal{G}$ (not containing the empty set) of $\wp(X)$ is called a grill on X if $\mathcal{G}$ satisfies the following conditions:

1. $A \in \mathcal{G}$ and $A \subseteq B$ implies $B \in \mathcal{G}$;

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2. $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Example 2.4. Let X be a nonempty set. Then the filter $\mathcal{F}$ (Thron, 1966) and the grill (Choquet, 1947) do not form a minimal structure on X.

Definition 2.5 (Alimohammady and Roohi, 2005) A set $A \in \wp(X)$ is said to be an $m-$open set if $A \in \mathcal{M}$. $B \in \wp(X)$ is an $m-$closed set if $X \setminus B \in \mathcal{M}$. We set

$$m - Int(A) = \cup\{U : U \subseteq A, U \in \mathcal{M}\},$$

$$m - Cl(A) = \cap\{F : A \subseteq F, X \setminus F \in \mathcal{M}\}.$$

The Proposition 2.6 of Alimohammady and Roohi (2005) can be restated by the following:

Proposition 2.6. *Let $(X, \mathcal{M})$ be a minimal space. Then for $A, B \in \wp(X)$,*

1. $m - Int(A) \subseteq A$ *and* $m - Int(A) = A$ *if* A *is an* $m-$*open set.*
2. $A \subseteq m - Cl(A)$ and $A = m - Cl(A)$ *if* A *is an* $m-$*closed set.*
3. $m - Int(A) \subseteq m - Int(B)$ *and* $m - Cl(A) \subseteq m - Cl(B)$ if $A \subseteq B$.
4. $m - Int(A \cap B) \subseteq (m - Int(A)) \cap (m - Int(B))$ *and* $(m - Int(A)) \cup (m - -Int(B)) \subseteq m - Int(A \cup B)$.
5. $m - Cl(A \cup B) \subseteq (m - Cl(A)) \cup (m - Cl(B))$ *and* $m - Cl(A \cap B) \subseteq (m - Cl(A)) \cap (m - Cl(B))$.
6. $m - Int(m - Int(A)) = m - Int(A)$ *and* $m - Cl(m - Cl(B)) = m - Cl(B)$.
7. $x \in m - Cl(A)$ *if and only if every* $m-$*open set* U_x *containing* $x, U_x \cap A \neq \emptyset$.
8. $(X \setminus m - Cl(A)) = m - Int(X \setminus A)$ *and* $(X \setminus m - Int(A)) = m - Cl(X \setminus A)$.

Definition 2.7 (Alimohammady and Roohi, 2005). A minimal space $(X, \mathcal{M})$ enjoys the property I if the finite intersection of m – open sets is an m – open set.

Example 2.8. Let X be a nonempty set. Let M be the $m-$structure (see Al-Omari and Noiri, 2012a) on X. Then the space (X, M) is an example of a minimal space with the property I.

A minimal space $(X, \mathcal{M})$ with grill $\mathcal{G}$ on X is called a grill minimal space and denoted as $(X, \mathcal{M}, \mathcal{G})$.

3. $()^{*\mathcal{M}}$-operator

In this section we obtain a topology from the minimal structure and the grill.

Definition 3.1. Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space. A mapping $()^{*\mathcal{M}} : \wp(X) \to \wp(X)$ is defined as follows:

$(A)^{*\mathcal{M}} = (A)^{*\mathcal{M}\mathcal{G}} = \{x \in X : A \cap U \in \mathcal{G}$ for all $U \in \mathcal{M}(x)\}$ for each $A \in \wp(X)$, where $\mathcal{M}(x) = \{U \in \mathcal{M} : x \in U\}$.

The mapping $()^{*\mathcal{M}}$ is called $\mathcal{M}$-local function.

3.1. Properties of $()^{*\mathcal{M}}$-operator

Here we have divided the properties of $()^{*\mathcal{M}}$ into two parts. Some of the properties hold at grill minimal space and other properties hold at minimal space with property I.

Theorem 3.2. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space. Then*

1. $(\emptyset)^{*\mathcal{M}} = \emptyset$.
2. $(A)^{*\mathcal{M}} = \emptyset$, *if* $A \notin \mathcal{G}$.
3. *for* $A, B \in \wp(X)$ *and* $A \subseteq B, (A)^{*\mathcal{M}} \subseteq (B)^{*\mathcal{M}}$.
4. *for* $A \subseteq X, (A)^{*\mathcal{M}} \subseteq m - Cl(A)$.
5. *for* $A \subseteq X, m - Cl[(A)^{*\mathcal{M}}] \subseteq (A)^{*\mathcal{M}}$.
6. *for* $A \subseteq X, (A)^{*\mathcal{M}}$ *is an* $m-$*closed set.*
7. *for* $A \subseteq X, [(A)^{*\mathcal{M}}]^{*\mathcal{M}} \subseteq (A)^{*\mathcal{M}}$.
8. $(A)^{*\mathcal{M}\mathcal{G}} \subseteq (A)^{*\mathcal{M}\mathcal{G}_1}$, *where* $\mathcal{G}_1$ *is a grill on* X *with* $\mathcal{G} \subseteq \mathcal{G}_1$.

Proof.

(1) Obvious from definition of $()^{*\mathcal{M}}$.
(2) Obvious from definition of $()^{*\mathcal{M}}$.
(3) Let $x \in (A)^{*\mathcal{M}}$. Then for all $U \in \mathcal{M}(x), U \cap A \in \mathcal{G}$. Again it is obvious that $U \cap B \in \mathcal{G}$ (from definition of grill). Hence $x \in (B)^{*\mathcal{M}}$.
(4) Let $x \notin m - Cl(A)$, then from Proposition 2.6, there is an U_x such that $U_x \cap A = \phi \notin \mathcal{G}$. Implies that $x \notin (A)^{*\mathcal{M}}$. Hence $(A)^{*\mathcal{M}} \subseteq m - Cl(A)$.
(5) Let $x \in m - Cl[(A)^{*\mathcal{M}}]$ and $U \in \mathcal{M}(x)$, then $U \cap (A)^{*\mathcal{M}} \neq \emptyset$. Let $y \in U \cap (A)^{*\mathcal{M}}$ Then $y \in U$ and $y \in (A)^{*\mathcal{M}}$. Therefore $U \cap A \in \mathcal{G}$, and hence $x \in (A)^{*\mathcal{M}}$. Thus $m - Cl[(A)^{*\mathcal{M}}] \subseteq (A)^{*\mathcal{M}}$.
(6) Proof is obvious from Proposition 2.6 and above Property.
(7) From Property 4, $[(A)^{*\mathcal{M}}]^{*\mathcal{M}} \subseteq m - Cl[(A)^{*\mathcal{M}}]$. Again from Property 5,$m - Cl[(A)^{*\mathcal{M}}] \subseteq (A)^{*\mathcal{M}}$. So, $[(A)^{*\mathcal{M}}]^{*\mathcal{M}} \subseteq (A)^{*\mathcal{M}}$.
(8) Obvious from definition grill. □

The Authors, Roy, Muhkerjee, Al-Omari, Noiri, Hatir and Jafiri have considered grill topological space for above theorem. But we have shown that the grill minimal space is the sufficient space for the same.

Second type properties of $()^{*\mathcal{M}}$-operator are:

Theorem 3.3. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space and $(X, \mathcal{M})$ enjoys the property I. Then*

1. *for* $A, B \subseteq X, (A \cup B)^{*\mathcal{M}} = (A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}}$.
2. *for* $U \in \mathcal{M}$ *and* $A \subseteq X, U \cap (A)^{*\mathcal{M}} = U \cap (U \cap A)^{*\mathcal{M}}$.
3. *for* $A, B \subseteq X, [(A)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}] = [(A \setminus B)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}]$.
4. *for* $A, B \subseteq X$ *with* $B \notin \mathcal{G}, (A \cup B)^{*\mathcal{M}} = (A)^{*\mathcal{M}} = (A \setminus B)^{*\mathcal{M}}$.

Proof.

(1) From Theorem 3.2(3), $(A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}} \subseteq (A \cup B)^{*\mathcal{M}}$. For reverse inclusion, suppose that $x \notin (A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}}$. Then there are $U_1, U_2 \in \mathcal{M}(x)$ such that $U_1 \cap A \notin \mathcal{G}, U_2 \cap B \notin \mathcal{G}$ and hence $(U_1 \cap A) \cup (U_2 \cap B) \notin \mathcal{G}$. Now $U_1 \cap U_2 \in \mathcal{M}(x)$ and $(A \cup B) \cap (U_1 \cap U_2) \subseteq (U_1 \cap A) \cup (U_2 \cap B) \notin \mathcal{G}$, so, $x \notin (A \cup B)^{*\mathcal{M}}$. Therefore $(A \cup B)^{*\mathcal{M}} \subseteq (A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}}$. Hence the result.
(2) From Theorem 3.2(3), $U \cap (U \cap A)^{*\mathcal{M}} \subseteq U \cap (A)^{*\mathcal{M}}$. For reverse inclusion, suppose $x \in U \cap (A)^{*\mathcal{M}}$ and $V \in \mathcal{M}(x)$. Then $U \cap V \in \mathcal{M}(x)$ and $x \in (A)^{*\mathcal{M}}$, implies $(U \cap V) \cap A \in \mathcal{G}$. So $(U \cap A) \cap V \in \mathcal{G}$. This implies that $x \in (U \cap A)^{*\mathcal{M}}$. Thus $x \in U \cap (U \cap A)^{*\mathcal{M}}$.

(3) Here, $(A)^{*\mathcal{M}} = [(A \setminus B) \cup (A \cap B)]^{*\mathcal{M}} = [(A \setminus B)^{*\mathcal{M}} \cup (A \cap B)^{*\mathcal{M}}]$ (from Theorem 3.3 (1)) $\subseteq [(A \setminus B)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}}]$ (from Theorem 3.2(3)). Thus $[(A)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}] \subseteq [(A \setminus B)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}]$. Again, $(A \setminus B)^{*\mathcal{M}} \subseteq (A)^{*\mathcal{M}}$ (from Theorem 3.2(3)). This implies that $[(A \setminus B)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}] \subseteq [(A)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}]$. Hence $[(A)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}} = [(A \setminus B)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}]$.

(4) From Theorem 3.3(1), $(A \cup B)^{*\mathcal{M}} = (A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}} = (A)^{*\mathcal{M}}$ (from Theorem 3.2(2)). Again from Theorem 3.2 (3), $(A \setminus B)^{*\mathcal{M}} \subseteq (A)^{*\mathcal{M}}$. Also from Theorem 3.3(3), $[(A)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}] \subseteq (A \setminus B)^{*\mathcal{M}}$. This implies that $(A)^{*\mathcal{M}} \subseteq (A \setminus B)^{*\mathcal{M}}$, since $B \notin \mathcal{G}$. Thus $(A)^{*\mathcal{M}} = (A \setminus B)^{*\mathcal{M}}$. □

Now we shall give an example, which shows that the condition *I* on grill minimal space is an essential condition for the above theorem.

Example 3.4. Let $X = \{a,b,c,d\}, \mathcal{M} = \{\emptyset, X, \{c\}, \{d\}, \{c,d\}, \{a,c\}, \{b,c\}, \{a,d\}, \{b,d\}, \{a,b,c\}, \{b,c,d\}, \{a,c,d\}, \{a,b,d\}\}$ and $\mathcal{G} = \wp(X) \setminus \{\emptyset\}$. Then the space $(X, \mathcal{M})$ does not enjoy the property *I*. Now, $\mathcal{M}$ open sets containing a are: $X, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,c,d\}, \{a,b,d\}; \mathcal{M}$ open sets containing b are: $X, \{b,c\}, \{b,d\}, \{a,b,c\}, \{b,c,d\}, \{a,b,d\}; \mathcal{M}$ open sets containing c are: $X, \{c\}, \{c,d\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, \{a,c,d\}; \mathcal{M}$ open sets containing d are: $X, \{d\}, \{c,d\}, \{a,d\}, \{b,d\}, \{b,c,d\}, \{a,c,d\}, \{a,b,d\}$. Consider $A = \{c\}, B = \{d\}$, then $(A)^{*\mathcal{M}} = \{c\}, (B)^{*\mathcal{M}} = \{d\}$. Now $(A \cup B)^{*\mathcal{M}} = \{a,b,c,d\}$. Hence $(A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}} \neq (A \cup B)^{*\mathcal{M}}$.

Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space. We define a map $Cl_{\mathcal{MG}} : \wp(X) \to \wp(X)$ by $Cl_{\mathcal{MG}}(A) = A \cup (A)^{*\mathcal{M}}$, for all $A \in \wp(X)$. Then we have:

Theorem 3.5. *The above map $Cl_{\mathcal{MG}}$ satisfies Kuratowski Closure axioms.*

Proof. From Theorem 3.2, $Cl_{\mathcal{MG}}(\emptyset) = \emptyset$, and obviously $A \subseteq Cl_{\mathcal{MG}}(A)$. Now $Cl_{\mathcal{MG}}(A \cup B) = (A \cup B) \cup (A \cup B)^{*\mathcal{M}} = (A \cup B) \cup (A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}}$ (from Theorem 3.3(1)) $= Cl_{\mathcal{MG}}(A) \cup Cl_{\mathcal{MG}}(B)$. Again for any $A \subseteq X, Cl_{\mathcal{MG}} [Cl_{\mathcal{MG}}(A)] = Cl_{\mathcal{MG}} [A \cup (A)^{*\mathcal{M}}] = [A \cup (A)^{*\mathcal{M}}] \cup [A \cup (A)^{*\mathcal{M}}]^{*\mathcal{M}} = A \cup (A)^{*\mathcal{M}} \cup [(A)^{*\mathcal{M}}]^{*\mathcal{M}}$ (from Theorem 3.3(1)) $= A \cup (A)^{*\mathcal{M}}$ (from Theorem 3.2(7) $= Cl_{\mathcal{MG}}(A)$. □

If $\mathcal{G}$ is a grill on X and $(X, \mathcal{M})$ is a minimal space enjoys the property *I*, then from Kuratowski Closure operator $Cl_{\mathcal{MG}}$, we get an unique topology on X which is given by following:

Theorem 3.6. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space and $(X, \mathcal{M})$ enjoys the property I. Then $\tau_{\mathcal{MG}} = \{V \subseteq X : Cl_{\mathcal{MG}}(X \setminus V) = X \setminus V\}$ is a topology on X, where $Cl_{\mathcal{MG}}(A) = A \cup (A)^{*\mathcal{M}}$.*

3.2. Properties of the topology $\tau_{\mathcal{MG}}$

Theorem 3.7.

(a) *If* $\mathcal{G}_1$ and $\mathcal{G}_2$ are two grills on X with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, *then* $\tau_{\mathcal{MG}2} \subseteq \tau_{\mathcal{MG}1}$.

(b) *If $\mathcal{G}$ is a grill on a set X and $B \notin \mathcal{G}$, then B is closed in $(X, \tau_{\mathcal{MG}})$.*

(c) *For any $A \subseteq X$, $(A)^{*\mathcal{M}}$ is $\tau_{\mathcal{MG}}$-closed.*

Proof.

(a) Let $U \in \tau_{\mathcal{MG}2}$. Then $Cl_{\mathcal{MG}2}(X \setminus U) = (X \setminus U) \cup (X \setminus U)^{*\mathcal{MG}2}$. Thus $(X \setminus U)^{*\mathcal{MG}2} \subseteq (X \setminus U)$, since $X \setminus U$ is $\tau_{\mathcal{MG}}$ closed set. Implies that $(X \setminus U)^{*\mathcal{MG}1} \subseteq (X \setminus U)$ (from Theorem 3.2(8)). So $(X \setminus U) = Cl_{\mathcal{MG}1}(X \setminus U)$, and hence $U \in \tau_{\mathcal{MG}1}$.

(b) It is obvious that, for $B \notin \mathcal{G}, (B)^{*\mathcal{M}} = \phi$. Then $Cl_{\mathcal{MG}}(B) = B \cup (B)^{*\mathcal{M}} = B$. Hence B is $\tau_{\mathcal{MG}}$-closed.

(c) We have, $Cl_{\mathcal{MG}}((A)^{*\mathcal{M}}) = (A)^{*\mathcal{M}} \cup ((A)^{*\mathcal{M}})^{*\mathcal{M}} = (A)^{*\mathcal{M}}$. Thus $(A)^{*\mathcal{M}}$ is $\tau_{\mathcal{MG}}$-closed. □

Here we find a simple open base for the topology $\tau_{\mathcal{MG}}$ on X.

Theorem 3.8. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space. Then $\beta(\mathcal{M}, \mathcal{G}) = \{V \setminus A : V \in \mathcal{M}$ and $A \notin \mathcal{G}\}$ is an open base for $\tau_{\mathcal{MG}}$.*

Proof. Let $U \in \tau_{\mathcal{MG}}$ and $x \in U$. Then $(X \setminus U)$ is $\tau_{\mathcal{MG}}$ - closed so that $Cl_{\mathcal{MG}}(X \setminus U) = (X \setminus U)$, and hence $(X \setminus U)^{*\mathcal{M}} \subseteq (X \setminus U)$. Then $x \notin (X \setminus U)$ and so there exists $V \in \mathcal{M}(x)$ such that $(X \setminus U) \cap V \notin \mathcal{G}$. Let $A = (X \setminus U) \cap V$, then $x \notin A$ and $A \notin \mathcal{G}$. Thus $x \in (V \setminus A) = V \setminus [(X \setminus U) \cap V] = V \setminus (V \setminus U) \subseteq U, V \setminus A \in \beta(\mathcal{M}, \mathcal{G})$. It now suffices to observe that $\beta(\mathcal{M}, \mathcal{G})$ is closed under finite intersections. Let $V_1 \setminus A$, $V_2 \setminus B \in \beta(\mathcal{M}, \mathcal{G})$, that is $V_1, V_2 \in \mathcal{M}$ and $A, B \notin \mathcal{G}$. Then $V_1 \cap V_2 \in \mathcal{M}$ and $A \cup B \notin \mathcal{G}$. Now, $(V_1 \setminus A) \cap (V_2 \setminus B) = (V_1 \cap V_2) \setminus (A \cup B) \in \beta(\mathcal{M}, \mathcal{G})$, proving ultimate that $\beta(\mathcal{M}, \mathcal{G})$ is an open base for $\tau_{\mathcal{MG}}$. □

Corollary 3.9. *For any grill $\mathcal{G}$ and any minimal structure $\mathcal{M}$ on $X, \mathcal{M} \subseteq \beta(\mathcal{M}, \mathcal{G}) \subseteq \tau_{\mathcal{MG}}$.*

4. $\psi_{\mathcal{M}}$-operator

In this section we shall introduce another operator on grill minimal space. We shall also discuss the properties of the same operator in the front of the topology which has been obtained in the previous section. At first we shall give the definition:

Definition 4.1. Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space. An operator $\psi_{\mathcal{M}} : \wp(X) \to \mathcal{M}$ is defined as follows for every $A \in \wp(X), \psi_{\mathcal{M}}(A) = \psi_{\mathcal{MG}}(A) = \{x \in X$: there exists a $U \in \mathcal{M}(x)$ such that $U \setminus A \notin \mathcal{G}\}$ and observe that $\psi_{\mathcal{M}}(A) = X \setminus (X \setminus A)^{*\mathcal{M}}$.

The properties of $\psi_{\mathcal{M}}$-Operator has two types. One type of property holds in grill minimal space. Another type of property holds in restricted minimal space.

Theorem 4.2. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space. Then following properties hold*:

1. *If $A \subseteq X$, then $\psi_{\mathcal{M}}(A)$ is $\mathcal{M}$-open.*
2. *If $A \subseteq B$, then $\psi_{\mathcal{M}}(A) \subseteq \psi_{\mathcal{M}}(B)$.*

3. *If* $U \in \tau_{\mathcal{MG}}$, *then* $U \subseteq \psi_{\mathcal{M}}(U)$.
4. *If* $A \subseteq X$, *then* $\psi_{\mathcal{M}}(A) \subseteq \psi_{\mathcal{M}}(\psi_{\mathcal{M}}(A))$.
5. *Let* $A \subseteq X$, *then* $\psi_{\mathcal{M}}(A) = \psi_{\mathcal{M}}(\psi_{\mathcal{M}}(A))$ *if and only if* $(X \setminus A)^{*\mathcal{M}} = [(X \setminus A)^{*\mathcal{M}}]^{*\mathcal{M}}$.

Proof.

1. Obvious from definition.
2. Obvious from Theorem 3.2(7).
3. If $U \in \tau_{\mathcal{MG}}$, then $X \setminus U$ is $\tau_{\mathcal{MG}}$-closed which implies $(X \setminus U)^{*\mathcal{M}} \subseteq (X \setminus U)$ and hence $U \subseteq [X \setminus (X \setminus U)^{*\mathcal{M}}] = \psi_{\mathcal{M}}(U)$.
4. This follows from (1) and (3).
5. This follows from the facts:
   (i) $\psi_{\mathcal{M}}(A) = X \setminus (X \setminus A)^{*\mathcal{M}}$.
   (ii) $\psi_{\mathcal{M}}(\psi_{\mathcal{M}}(A)) = [X \setminus [X \setminus (X \setminus (X \setminus A)^{*\mathcal{M}})]^{*\mathcal{M}}] = [X \setminus [(X \setminus A)^{*\mathcal{M}}]^{*\mathcal{M}}]$. □

Corollary 4.3. *Let* $(X, \mathcal{M}, \mathcal{G})$ *be a grill minimal space. Then* $U \subseteq \psi_{\mathcal{M}}(U)$ *for every* $U \in \mathcal{M}$.

Theorem 4.4. *Let* $(X, \mathcal{M}, \mathcal{G})$ *be a grill minimal space and* $(X, \mathcal{M})$ *enjoys the property I. Then*

1. *If* $A, B \in \wp(X)$, *then* $\psi_{\mathcal{M}}(A \cap B) = \psi_{\mathcal{M}}(A) \cap \psi_{\mathcal{M}}(B)$.
2. *If* $A \notin \mathcal{G}$, *then* $\psi_{\mathcal{M}}(A) = X \setminus (X)^{*\mathcal{M}}$.
3. *If* $A \subseteq X$, *then* $A \cap \psi_{\mathcal{M}}(A) = int_{\mathcal{MG}}(A)$ (*where* $int_{\mathcal{MG}}(A)$ *denote the interior operator of* $(X, \tau_{\mathcal{MG}})$).
4. *If* $A \subseteq X$ *and* $G \notin \mathcal{G}$, *then* $\psi_{\mathcal{M}}(A \setminus G) = \psi_{\mathcal{M}}(A)$.
5. *If* $A \subseteq X$ *and* $G \notin \mathcal{G}$, *then* $\psi_{\mathcal{M}}(A \cup G) = \psi_{\mathcal{M}}(A)$.
6. *If* $A, B \in \wp(X)$ *and* $(A \setminus B) \cup (B \setminus A) \notin \mathcal{G}$, *then* $\psi_{\mathcal{M}}(A) = \psi_{\mathcal{M}}(B)$.

Proof.

1. It is obvious from Theorem 4.2(2), $\psi_{\mathcal{M}}(A \cap B) \subseteq \psi_{\mathcal{M}}(A)$ and $\psi_{\mathcal{M}}(A \cap B) \subseteq \psi_{\mathcal{M}}(B)$. Hence $\psi_{\mathcal{M}}(A \cap B) \subseteq \psi_{\mathcal{M}}(A) \cap \psi_{\mathcal{M}}(B)$. Now, let $x \in \psi_{\mathcal{M}}(A) \cap \psi_{\mathcal{M}}(B)$. Then there exists $U, V \in \mathcal{M}(x)$ such that $U \setminus A \notin \mathcal{G}$ and $V \setminus B \notin \mathcal{G}$. Let $G = U \cap V \in \mathcal{M}(x)$ and we have $G \setminus A \notin \mathcal{G}$ and $G \setminus B \notin \mathcal{M}$ (from definition of $\psi_{\mathcal{M}}$-operator). Thus $[G \setminus (A \cap B)] = [(G \setminus A) \cup (G \setminus B)] \in \mathcal{G}$ (from definition of grill), and hence $x \in \psi_{\mathcal{M}}(A \cap B)$. We have shown that $\psi_{\mathcal{M}}(A) \cap \psi_{\mathcal{M}}(B) \subseteq \psi_{\mathcal{M}}(A \cap B)$. Hence the prove is completed.
2. We know from Theorem 3.3(1), $(X \setminus A)^{*\mathcal{M}} = (X)^{*\mathcal{M}}$ if $A \notin \mathcal{G}$. Then $\psi_{\mathcal{M}}(A) = X \setminus (X)^{*\mathcal{M}}$.
3. If $x \in A \cap \psi_{\mathcal{M}}(A)$, then $x \in A$ and there exists a $U \in \mathcal{M}(x)$ such that $U \setminus A \notin \mathcal{G}$. Then by Theorem 3.8, $[U \setminus (U \setminus A)]$ is a $\tau_{\mathcal{MG}}$-open neighborhood of x and $x \in int_{\mathcal{MG}}(A)$. Conversely suppose that $x \in int_{\mathcal{MG}}(A)$, there exists a basic $\tau_{\mathcal{MG}}$-open neighborhood $V \setminus G$ of x where $V \in \mathcal{M}(x)$ and $G \notin \mathcal{G}$, such that $x \in V \setminus G \subseteq A$ which implies that $V \setminus A \subseteq G$ and hence $V \setminus A \notin \mathcal{G}$. Hence $x \in A \cap \psi_{\mathcal{M}}(A)$.
4. $\psi_{\mathcal{M}}(A \setminus G) = [X \setminus [X \setminus (A \setminus G)]^{*\mathcal{M}}] = [X \setminus [(X \setminus A) \cup G]^{*\mathcal{M}}] = [X \setminus (X \setminus A)^{*\mathcal{M}}]$(since $G \notin \mathcal{G}$) $= \psi_{\mathcal{M}}(A)$.
5. $\psi_{\mathcal{M}}(A \cup G) = X \setminus [X \setminus (A \cup G)]^{*\mathcal{M}} = X \setminus [(X \setminus A) \setminus G)]^{*\mathcal{M}} = X \setminus (X \setminus A)^{*\mathcal{M}}$ (from 4) $= \psi_{\mathcal{M}}(A)$.
6. Assume $(A \setminus B) \cup (B \setminus A) \notin \mathcal{G}$. Let $A \setminus B = G_1$ and $B \setminus A = G_2$. Observe that $G_1, G_2 \notin \mathcal{G}$ (from definition of grill). Also observe that $B = (A \setminus G_1) \cup G_2$. Thus $\psi_{\mathcal{M}}(A) = \psi_{\mathcal{M}}(A \setminus G_1) = \psi_{\mathcal{M}}[(A \setminus G_1) \cup G_2] = \psi_{\mathcal{M}}(A)$ (from (4) and (5)). □

Now we shall show that the property I is an essential condition for the previous theorem.

Example 4.5. Here we consider the Example 3.4. Let $A = \{a, b, d\}$ and $B = \{a, b, c\}$. Then $\psi_{\mathcal{M}}(A) = X \setminus (\{c\})^{*\mathcal{M}} = X \setminus \{c\} = \{a, b, d\}$ and $\psi_{\mathcal{M}}(B) = X \setminus (\{d\})^{*\mathcal{M}} = X \setminus \{d\} = \{a, b, c\}$. Hence we have:

$$\psi_{\mathcal{M}}(A \cap B) \neq \psi_{\mathcal{M}}(A) \cap \psi_{\mathcal{M}}(B).$$

Therefore we conclude that the topological space is not the only space for discussing the properties of $\psi_{\mathcal{M}}$-Operator. Al-Omari and Noiri (2012a) is also a suitable space for the same. Moreover grill minimal space is also a suitable space.

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