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Characterization of Burr-Type X Distribution Based on Conditional Expectations of Dual Generalized Order Statistics

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Abstract: In this paper, the characterization of Burr -Type X distribution by conditional expectation of function of dual generalized order statistics based on non-adjacent dual generalized order statistics are obtained. Further, the results of reversed order statistic, order statistic and lower record values are discussed.

Keywords: Dual generalized order statistics, Reversed order statistic, Order statistic, Lower record statistic, Conditional expectations and characterization.

1. Introduction

Kamps [12] introduced the concept of the generalized order statistics (gos) as a general framework models of ordered random variables. Sequential order statistic, upper record statistic, progressively Type-II censored order statistics and some other ordered random variables can be considered as a special cases of the gos. These models can be effectively applied, e.g., in reliability theory. Although the gos contains many useful models of ordered random variables, the random variables that are decreasingly ordered cannot be integrated into this frame. Consequently, this model is inappropriate to study. Using the concept of gos., Burkschat *et al*. [7] introduced the concept of dual generalized order statistics $(dgos)$ as a systematic approach to some models of decreasingly ordered random variables.

Dual generalized order statistics represents a unification of models of decreasingly ordered random variables e.g., reversed order statistic, lower records, lower k- records and lower Pfeifer records.

Let $F(x)$ be an absolutely continuous distribution function (df) with the probability density function (pdf) $f(x)$ Further, let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$, $\forall$ $r \in \{1, 2, \ldots, n-1\}$. Then, $X'(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called $(dgos)$ if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m_i} f(x_i)\right)[F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

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for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

Throughout the paper, we assume $m_i = m_j = m$, $i, j = 1,2,\ldots,n-1$.

The pdf of the r^{th} $dgos$ is given by,

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)). \tag{1.2}$$

The joint pdf of the r^{th} and s^{th} $dgos$ is given by,

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x)\, g_m^{r-1} F(x)$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s - 1} f(y), \quad \alpha \leq y < x \leq \beta, \tag{1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1} x^{m+1} & , m \neq -1 \\ -\log x & , m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \qquad x \in [0,1).$$

An absolutely continuous random variable (rv) X has a Burr -Type X distribution if its cumulative distribution function (cdf) $F(x)$ is as follows,

$$F(x) = [1 - e^{-\beta x^2}]^{\alpha}, \quad x \geq 0, \ (\alpha > 0, \beta > 0), \tag{1.4}$$

where α, β are shape and scale parameters respectively.

The corresponding pdf $f(x)$ is given by,

$$f(x) = 2\alpha\beta x e^{-\beta x^2} (1 - e^{-\beta x^2})^{\alpha - 1}, \quad x \geq 0, \ (\alpha > 0, \beta > 0). \tag{1.5}$$

The above distribution is member of the family of Burr distribution which appeared in 1942, Burr [4].

The pdf of Burr -Type X distribution can take different shapes. It is a right skewed unimodal function for $\alpha > \frac{1}{2}$ and is a decreasing function of x if $\alpha \leq \frac{1}{2}$. For the failure rate of Burr distribution, it has an increasing failure rate at $\alpha > \frac{1}{2}$ and it is bathtub type when $\alpha \leq \frac{1}{2}$.

It is found that this distribution fits many practical data due its flexibility, in several areas of applications such as lifetime tests, health, agriculture, biology and social sciences. For more details, see (Ahmad *et al*.[1], Raqab and Kundu [11] and Surles and Padgett [5]).

Characterization of probability distributions play an important role in probability and statistics. Different methods are used for the characterization of continuous distributions . Characterization based on conditional expectations is one of them. see, for example, Khan et al.[2] among others.

Conditional expectations of dual generalized order statistics are extensively used in, characterizing of distributions. For detailed survey and discussion of characterization results through $dgos$ one may refer to Ahsanullah [6], Mbah and Ahasanullah [3], Faizan and Khan [8], Tavanagar [10] and Khan and Faizan [9].

It appears from literature that no attention has been paid on the characterization of Burr –Type X distribution through conditional expectations of dual generalized order statistic.

In this paper, we present a characterization of Burr-Type X distribution through conditional expectations of function of dual generalized order statistics conditioned on non-adjacent $dgos$.

2. CHARACTERIZATION OF DISTRIBUTION

Let $X'(r,n,m,k)$, $r=1,2,\dots,n$ be r^{th} $dgos$, then the conditional pdf of $X'(s,n,m,k)$ given $X'(r,n,m,k)=x,\ 1\le r<s\le n$, in view of (1.2) and (1.3) is,

$$f_{s|r}(y\mid x)=\frac{C_{s-1}}{C_{r-1}(s-r-1)!}\frac{[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-1}[F(y)]^{\gamma_s-1}}{(m+1)^{s-r-1}[F(x)]^{\gamma_{r+1}}}f(y),\, y<x. \tag{2.1}$$

Theorem 2.1. Let $X'(r,n,m,k)$, $r=1,2,\dots,n$ be the r^{th} $dgos$ from a continuous population with the $df\ F(x)$ and the $pdf\ f(x)$. Then for $1\le r<s\le n$,

$$E[e^{-\beta X'^2(s,n,m,k)}\mid X'(l,n,m,k)=x]=a_{s|r}\,e^{-\beta x^2}+b_{s|r}\ ,l=r,r+1, \tag{2.2}$$

if and only if

$$F(x)=[1-e^{-\beta x^2}]^{\alpha},\ x\ge 0,\ (\alpha>0,\beta>0), \tag{2.3}$$

where $a_{s|r}=\prod_{j=r+1}^{s}\frac{\alpha\gamma_j}{1+\alpha\gamma_j}$ and $b_{s|r}=[1-a_{s|r}]$.

Proof. We have,

$$E[e^{-\beta X'^2(s,n,m,k)}\mid X'(r,n,m,k)=x]$$

$$=\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_0^x e^{-\beta y^2}\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}$$

$$\times\left[\frac{F(y)}{F(x)}\right]^{k+(m+1)(n-s)-1}\frac{f(y)}{F(x)}dy. \tag{2.4}$$

Set

$$\left[\frac{F(y)}{F(x)}\right]^{m+1} = \left[\frac{1-e^{-\beta y^2}}{1-e^{-\beta x^2}}\right]^{\alpha(m+1)} = u .$$

Then the R.H.S. of (2.4) reduces to,

$$E[e^{-\beta X'^2(s,n,m,k)} \mid X'(r,n,m,k)=x]$$

$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r}} \int_0^1 [1-\{u^{\frac{1}{\alpha(m+1)}}(1-e^{-\beta x^2})\}]\,(1-u)^{s-r-1} u^{\frac{\gamma_s-1}{m+1}-\frac{m}{m+1}} du .$$

Thus,

$$E[e^{-\beta X'^2(s,n,m,k)} \mid X'(r,n,m,k)=x] = 1-[1-e^{-\beta x^2}]\prod_{j=r+1}^{s}\frac{\alpha\gamma_j}{1+\alpha\gamma_j} ,$$

$$E[e^{-\beta X'^2(s,n,m,k)} \mid X'(r,n,m,k)=x] = a_{s|r}\, e^{-\beta x^2} + b_{s|r} ,$$

That is,

$$g_{s|r}(x) = a_{s|r}\, e^{-\beta x^2} + b_{s|r} ,$$

and hence the necessary part.

To prove the sufficiency part, we have (Khan *et al.*[2]),

if $E[e^{-\beta X'^2(s,n,m,k)} \mid X'(r,n,m,k)=x] = g_{s|r}(x)$,

then $\dfrac{f(x)}{F(x)} = \dfrac{1}{\gamma_{r+1}} \dfrac{g'_{s|r}(x)}{[g_{s|r+1}(x) - g_{s|r}(x)]}$.

Now,

$$g_{s|r+1}(x) - g_{s|r}(x) = \left(a_{s|r+1} - a_{s|r}\right)\left(e^{-\beta x^2} - 1\right) = \frac{a_{s|r}}{\alpha\gamma_{r+1}}\left(e^{-\beta x^2} - 1\right).$$

Therefore,

$$\frac{1}{\gamma_{r+1}} \frac{g'_{s|r}(x)}{[g_{s|r+1}(x) - g_{s|r}(x)]} = \frac{2\alpha\beta x e^{-\beta x^2}}{[1-e^{-\beta x^2}]} .$$

Thus,

$$\frac{f(x)}{F(x)} = \frac{2\alpha\beta x e^{-\beta x^2}}{[1-e^{-\beta x^2}]} ,$$

implying that

$$F(x) = [1-e^{-\beta x^2}]^{\alpha},\ x \ge 0,\ (\alpha > 0, \beta > 0) ,$$

and hence the Theorem.

Remark 2.1. Putting $m=0, k=1,$ in the Theorem 2.1, it reduces to reversed order statistic as follows,

$$E[e^{-\beta X'^2_{s:n}} \mid X'_{r:n}=x]=a_{s|r}e^{-\beta x^2}+b_{s|r},$$

where $a_{s|r}=\prod_{j=r+1}^{s}\frac{\alpha(n-j+1)}{\alpha(n-j+1)+1}$ and $b_{s|r}=[1-a_{s|r}]$.

And for order statistic, it will reduce to

$$E[e^{-\beta X^2_{n-s+1:n}} \mid X_{n-r+1:n}=x]=a_{s|r}e^{-\beta x^2}+b_{s|r},$$

as $X'_{r:n}=X_{n-r+1:n}$, where $X_{r:n}$ is the r^{th} order statistic.

Remark 2.2. As $m\to -1,$ in the Theorem 2.1, it reduces to lower record statistic as follows,

$$E[e^{-\beta X'^2_{L(s)}} \mid X_{L(r)}=x]=a_{s|r}e^{-\beta x^2}+b_{s|r},$$

where $a_{s|r}=\left(\frac{\alpha k}{1+\alpha k}\right)^{s-r}$ and $b_{s|r}=[1-a_{s|r}]$.

Theorem 2.2. Let $X'(r,n,m,k)$, $r=1,2,\ldots,n$ be the r^{th} $dgos$ from a continuous population with the df $F(x)$ and the pdf $f(x)$. Then for $1\le r<s<t\le n$,

$$E[e^{-\beta X'^2(t,n,m,k)} \mid X'(r,n,m,k)=x]$$
$$=a_{t|s}E[e^{-\beta X'^2(s,n,m,k)} \mid X'(r,n,m,k)=x]+b_{t|s}, \qquad (2.5)$$

if and only if

$$F(x)=[1-e^{-\beta x^2}]^{\alpha}, \quad x\ge 0, \ (\alpha>0, \beta>0), \qquad (2.6)$$

where $a_{t|s}=\prod_{j=s+1}^{t}\frac{\alpha\gamma_j}{1+\alpha\gamma_j}$ and $b_{t|s}=[1-a_{t|s}]$.

Proof. For the necessary part, we have (in view of Theorem 2.1),

$$E[e^{-\beta X'^2(t,n,m,k)} \mid X'(r,n,m,k)=x] = a_{t|r}e^{-\beta x^2}+b_{t|r}$$
$$=a_{t|r}\left(e^{-\beta x^2}-1\right)+1$$
$$=a_{t|s}a_{s|r}\left(e^{-\beta x^2}-1\right)+1$$
$$=a_{t|s}\left[a_{s|r}\left(e^{-\beta x^2}-1\right)+1\right]-a_{t|s}+1$$
$$=a_{t|s}[a_{s|r}e^{-\beta x^2}+b_{s|r}]+b_{t|s},$$

where,

$$a_{t|r} = \frac{\gamma_t}{\gamma_s} \times \frac{\gamma_s}{\gamma_r} = a_{t|s} a_{s|r} \text{ and } b_{t|r} = [1 - a_{t|r}].$$

That is,

$$E[e^{-\beta X'^2(t,n,m,k)} \mid X'(r,n,m,k) = x]$$

$$= a_{t|s} E[e^{-\beta X'^2(s,n,m,k)} \mid X'(r,n,m,k) = x] + b_{t|s}.$$

This proves the necessary part.

For the sufficiency part, we have

$$\frac{C_{t-1}}{C_{r-1}(t-r-1)!(m+1)^{t-r-1}[F(x)]^{\gamma_{r+1}}} \int_0^x e^{-\beta y^2} [F(x)^{m+1} - F(y)^{m+1}]^{t-r-1} [F(y)]^{\gamma_t - 1} f(y) dy$$

$$= a_{t|s} \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}[F(x)]^{\gamma_{r+1}}} \int_0^x e^{-\beta y^2} [F(x)^{m+1} - F(y)^{m+1}]^{s-r-1}$$

$$\times [F(y)]^{\gamma_s - 1} f(y)\, dy + b_{t|s}. \tag{2.7}$$

Differentiating $(s-r)$ times both the sides of (2.7) $w.r.t\ x$, we get,

$$\frac{C_{t-1}}{C_{s-1}(t-s-1)!(m+1)^{t-s-1}[F(x)]^{\gamma_{s+1}}}$$

$$\times \int_0^x e^{-\beta y^2} [F(x)^{m+1} - F(y)^{m+1}]^{t-s-1} [F(y)]^{\gamma_t - 1} f(y) dy = a_{t|s} e^{-\beta x^2} + b_{t|s},$$

i.e.,

$$g_{t|s}(x) = a_{t|s} e^{-\beta x^2} + b_{t|s}.$$

Thus, we get,

$$F(x) = [1 - e^{-\beta x^2}]^{\alpha},\ x \geq 0,\ (\alpha > 0, \beta > 0),$$

as proved in the Theorem 2.1.

Remark 2.3. At $s = r$ Theorem 2.2 reduces to Theorem 2.1.

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