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A Note on the Convexity of a Generalized Pearson Distribution

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Abstract: In this paper, we investigate the convexity of a generalized Pearson distribution introduced by Shakil et al. [16], known as Shakil-Kibria-Singh (SKS) distribution, in literature. Several related properties are also discussed.

Keywords: Concavity, Convex functions, Log-Convexity, Generalized Pearson Distribution

1. INTRODUCTION

Convex functions play a very significant role in many areas of mathematical and management sciences. They are especially important in the study of optimization problems, Calculus of variation and probability theory. A function $f : R \to R_+$ is called convex if on an interval $[a,b] \subset R$, if $\forall\ x_1, x_2 \in [a,b]$

$f[\lambda x_1 + (1-\lambda)x_2] \le \lambda f(x_1) + (1-\lambda) f(x_2)$, where $0 \le \lambda \le 1$.

Also, if the second derivative $f''(x)$ of the function $f(x)$ exists on an interval $[a,b] \subset R$, then a necessary and sufficient condition for the function $f(x)$ to be concave on the interval $[a,b] \subset R$ is that $f''(x) < 0$, $\forall\ x_1, x_2 \in [a,b]$. Also, the function $f(x)$ to be convex on the interval $[a,b] \subset R$ is that $f''(x) > 0$, $\forall\ x_1, x_2 \in [a,b]$. For details, see Webster [19, page 97].

A function $f : R \to R_+$ is called log-concave if the function $\ln(f)$ is concave. Similarly, the function f is called log-convex if the function $\ln(f)$ is convex.

The notions of log-concavity and log-convexity play major roles in various disciplines like economics, political science, biology, industrial engineering, social sciences, information theory and optimization. Various properties of the notion of log-concavity/log-convexity and their applications can be found in Riordan and Sappington [15], Laffont and Tirole [11], Lewis and Sappington [12], Maskin and Riley [13], Caplin and Nalebuff [5, 6], Mathews [14], and Borzadaran and Borzadaran [3].

Let X be a continuous random variable with density function $f(x)$ and cumulative density function $F(x)$ defined on an open interval (a,b) of the set of real numbers R then $\forall x \in R$, $S(x) = 1 - F(x)$ is called survivor function; $h(x) = \dfrac{f(x)}{S(x)}$ is called hazard function, $L(x) = \int_a^x F(t)dt$ is called Left side integral and $R(x) = \int_x^b S(t)dt$ is called right side integral of $f(x)$.

The log-concavity and log-convexity of survival function $S(x) = 1 - F(x)$ have applications in reliability theory and is used to determine the increase and decrease of failure rate. Jagadeesh and Chowdhry [10] has used log-concavity of reliability function in finance. Flinn and Heckman [7] has used the log-concavity of the function

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$$R(x) = \int_x^{\infty} S(t)dt,$$

to analyze the optimal strategy for jobs and unemployment. Similarly, Bagnoli and Bergstrom [1] have studied *marring market model* by using the formula

$$L(x) = \int_{-\infty}^{x} F(t)dt.$$

In this paper, we investigate the convexity of some probability models developed by Shakil et al. [16], known in the literature, as Shakil-Kibria-Singh (SKS) distribution. The organization of the paper is as follows. In Section 2, we briefly review SKS distribution. In Section 3, we investigate the convexity of SKS distribution. In Section 4, we examine the logarithmic convexity and infinite divisibility of SKS distribution. Finally, in Section 5, we provide some concluding remarks.

2. SKS DISTRIBUTION

A continuous distribution is said to belong to the Generalized Pearson systems if its pdf (probability density function) f satisfies a differential equation of the form

$$\frac{f'(x)}{f(x)} = \frac{\sum_{j=0}^{m} a_j x^j}{\sum_{j=0}^{n} b_j x^j}, \qquad (1.1)$$

where $a_j, b_j \in R$ are real numbers and $m, n \in Z_+$ are positive integers. The differential equation (1.2) is called the Generalized Pearson Differential Equation (GPDE). That is,

$$S = \left\{ f(x) \geq 0 : \frac{f'(x)}{f(x)} = \frac{\sum_{j=0}^{m} a_j x^j}{\sum_{j=0}^{n} b_j x^j}, a_j, b_j \in R; m, n \in Z_+ \right\}, \qquad (1.2)$$

where Z_+ is the set of all positive integers. The set *S* of probability density functions $f(x)$ is called the generalized Pearson System. Shakil et al. [16] obtained the solution of (1.2), by taking

$m = 2p, n = p+1, a_j = 0, j = 1, \cdots p-1, p+1, \cdots 2p-1; b_j = 0, j = 0,1, \cdots, p, b_{p+1} \neq 0 \text{ and } x > 0,$

Thus (1.1) resulted as

$\frac{f'(x)}{f(x)} = \frac{a_{2p}x^{2p} + a_p x^p + a_0}{b_{p+1}x^{p+1}}$, from which, following Shakil et al. [16], the following pdf is obtained

$$f(x) = Cx^{\nu-1}\exp(-\alpha x^p - \beta x^{-p}), \quad x > 0, \alpha, \beta \geq 0, \nu \in R, \qquad (1.3)$$

where $\alpha = -\frac{a_{2p}}{pb_{p+1}}$, $\beta = \frac{a_0}{pb_{p+1}}$, $\nu = \frac{a_p + b_{p+1}}{b_{p+1}}$, $b_{p+1} \neq 0$, $p \in Z_+$, and C is a normalizing constant which is classified as follows:

Case I: when $\alpha > 0$, $\beta > 0$, $\nu \in R$ and $p \in Z_+$, then the normalizing constant

$$C_1 = \left(\frac{p}{2}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{\nu}{2p}} \frac{1}{K_{\left\{\frac{\nu}{p}\right\}}\left(2\sqrt{\alpha\beta}\right)}, \text{ and } f(x) = C_1 x^{\nu-1} e^{-\alpha x^p - \beta x^{-p}} \qquad (1.4)$$

where $K\left(\frac{\nu}{p}\right)\left(2\sqrt{\alpha\beta}\right)$ represent the modified Bessel Function of third kind.

Case II: when $\alpha > 0,\ \beta = 0,\ \nu > 0$, and $p \in Z_+$; then the normalizing constant

$$C_2 = \frac{p\ (\alpha)^{\frac{\nu}{p}}}{\Gamma\left(\frac{\nu}{p}\right)}\ \text{, and}\ \ f(x) = C_2 x^{\nu-1} e^{-\alpha x^p} \tag{1.5}$$

Case III: **when** $\alpha = 0,\ \beta > 0,\ \nu < 0$, and $p \in Z_+$; then the normalizing constant

$$C_3 = \frac{p}{(\beta)^{\frac{\nu}{p}}\ \Gamma\left(-\frac{\nu}{p}\right)}\ \text{, and}\ f(x) = C_3\, x^{\nu-1} e^{-\beta x^{-p}} \tag{1.6}$$

Later, Hamedani [8, 9] characterized the above distribution and called it as Shakil-Kibria-Singh (SKS) distribution.

3. CONVEXITY OF SKS DISTRIBUTION

In this section we shall discuss the convexity of SKS distribution. For the details on the distributional properties of SKS distribution the interested readers are referred to Shakil et al. [16,17]

Case I: Differentiating the pdf (1.4) twice in case I, we have

$$\frac{d^2 f}{d x^2} = C_1 x^{\nu-3} e^{-\alpha x^p - \beta x^{-p}}$$

$$.\left[(\nu-1)(\nu-2) - 2\alpha\beta\ p^2 + \beta\, p(2\nu - p - 3)x^{-p} + \beta^2 p^2 x^{-2p} - \alpha\, p(2\nu + p - 3)x^p + \alpha^2 p^2 x^{2p}\right].$$

Thus, the pdf (1.4) will be convex if $\frac{d^2 f}{d x^2} \geq 0$, that is

$$\left[(\nu-1)(\nu-2) - 2\alpha\beta\ p^2 + \beta\, p(2\nu - p - 3)x^{-p} + \beta^2 p^2 x^{-2p} - \alpha\, p(2\nu + p - 3)x^p + \alpha^2 p^2 x^{2p}\right] \geq 0,$$

or

$$\left[\alpha^2 p^2 x^{4p} - \alpha\, p(2\nu + p - 3)x^{3p} + \left\{(\nu-1)(\nu-2) - 2\alpha\beta\ p^2\right\}x^{2p} + \beta\, p(2\nu - p - 3)x^p + \beta^2 p^2\right] \geq 0,$$

which needs to be solved numerically.

Example 2.1: When p =1, α = 4, β =2 and ν = 4, in case I, we have

$$f(x) = \frac{2}{K_4(4\sqrt{2})}\, x^3 e^{-4x - \frac{2}{x}},$$

$$\text{with}\ \ \frac{d^2 f}{d x^2} = (0.006443062951)\, x e^{-4x - \frac{2}{x}}\left(-10 + \frac{8}{x} + \frac{4}{x^2} - 24x + 16 * x^2\right).$$

Using Maple, the following graph of the above pdf has been drawn. It is evident from the figure 2.1 that the pdf is unimodal, skewed right and concave down, for the given values of parameters.

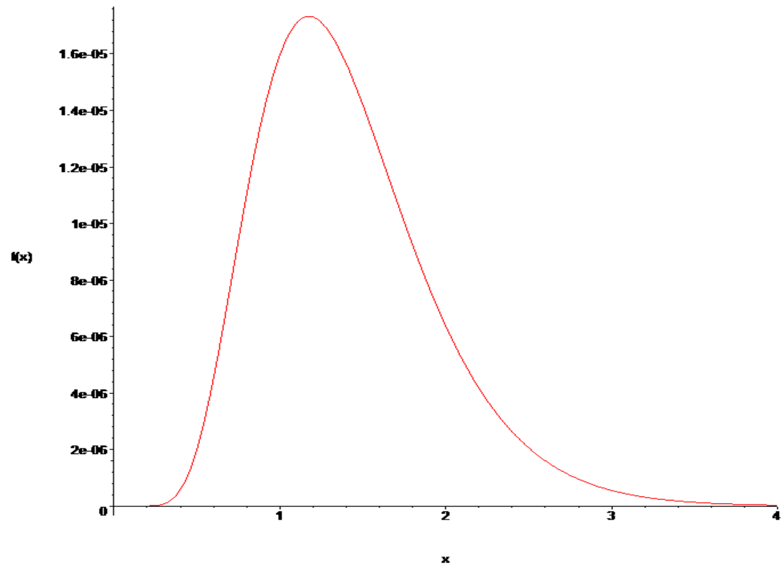

Figure 2.1 Graph of $f(x)$

After solving numerically using Maple, the following graphs of $f''(x)$ (in Figures 2.2 and 2.3) have been drawn in the intervals $(0, 0.715)$ and $(1.64, 4)$ respectively, for which $\frac{d^2 f}{d x^2} \geq 0$.

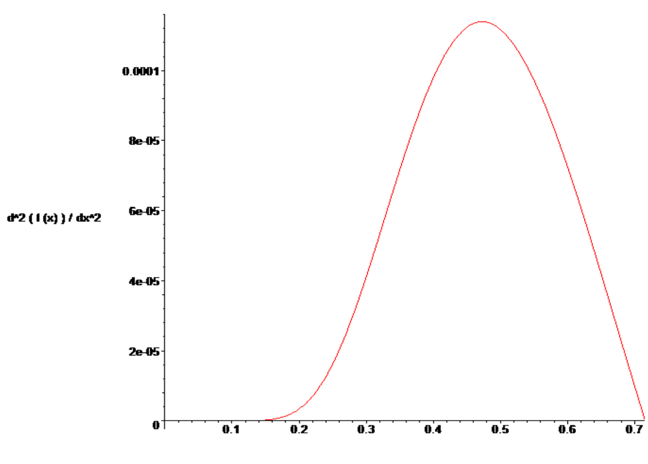

Figure 2.2 Graph of $f''(x)$ in (0, 0.715)

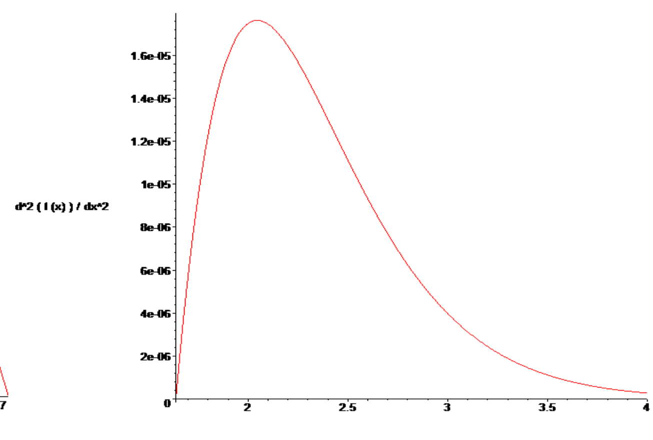

Figure 2.3 Graph of $f''(x)$ in (1.64, 4)

Consequently, the above pdf is convex in the intervals $(0, 0.715)$ and $(1.64, 4)$ respectively.

Case II: When $v = 1$ *and* $p > 1$ in case II, then differentiating pdf (1.5) twice, we have

$$f''(x) = C_2 \alpha\, p e^{-\alpha x^p} \left[\alpha p x^{2p-2} - (p-1) x^{p-2}\right].$$

Now $f''(x) < 0$, if $\left[\alpha p x^{2p-2} - (p-1)x^{p-2}\right] < 0$, that is, $x^{p-2}\left[\alpha p x^{p} - (p-1)\right] < 0$. This is possible only when $x^{p} < \left(\frac{p-1}{\alpha p}\right)$, that is, $x < \left(\frac{p-1}{\alpha p}\right)^{\frac{1}{p}}$.

So the density function $f(x)$ is concave, if $x \in \left[0, \left(\frac{p-1}{\alpha p}\right)^{\frac{1}{p}}\right]$.

Similarly, the density function $f(x)$ is convex, if $x \in \left[\left(\frac{p-1}{\alpha p}\right)^{\frac{1}{p}}, \infty\right)$.

In this case, the point $\left(\left(\frac{p-1}{\alpha p}\right)^{\frac{1}{p}}, f\left(\left(\frac{p-1}{\alpha p}\right)^{\frac{1}{p}}\right)\right)$ is the inflection point for the density function. In particular, when $p = 2$, then the density function $f(x)$ is concave when , $x \in \left(\frac{1}{\sqrt{2\alpha}}, \infty\right)$

and convex when $x \in \left(0, \frac{1}{\sqrt{2\alpha}}\right)$. The inflection point of the density is $\left(\frac{1}{\sqrt{2\alpha}}, f\left(\frac{1}{\sqrt{2\alpha}}\right)\right)$

Case III : Now considering the pdf (1.6) in case III, and differentiating it twice, we have

$$\frac{d^2 f}{d x^2} = C_3 x^{\nu-3} e^{-\beta x^{-p}} \left\{(\nu-1)(\nu-2) + \beta p (2\nu - p - 3) x^{-p} + \beta^2 p^2 x^{-2p}\right\}$$

Obviously, the pdf, in this case, will be convex if $\frac{d^2 f}{d x^2} \geq 0$, that is

$$\left\{(\nu-1)(\nu-2) + \beta p (2\nu - p - 3) x^{-p} + \beta^2 p^2 x^{-2p}\right\} \geq 0,$$

which needs to be solved numerically.

Example 2.2: Taking p = 3, β = 1, and ν = 4 in pdf (1.6), we have

$$f(x) = C_3\, x^3 e^{-x^{-3}}, \quad C_3 = \frac{3}{\Gamma\left(-\frac{4}{3}\right)} = 3.04676.$$

Using Maple, the graph of this density function is shown in the following figure.

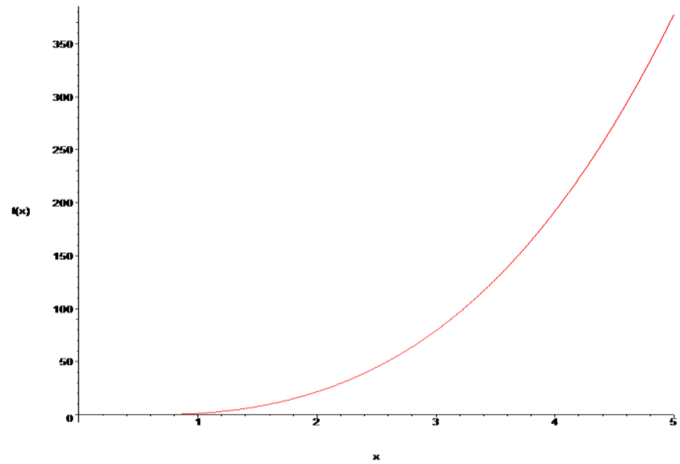

Figure 2.4 Graph of $f(x)$

It is evident from the figure 2.4 that the pdf, in this case, is skewed right and concave up for the specified values of parameters $p=3, \beta=1$, and ν = 4.
Further, we have

$$\frac{d^2 f}{d x^2} = 18.28056\, x\, e^{-\frac{1}{x^3}} \left\{1 + x^{-3} + 1.5 x^{-6}\right\} \geq 0, \quad \forall x > 0 .$$

Hence, in view of **Webster [19, page 197],** the above pdf is convex.

4. LOGARITHMIC CONVEXITY AND INFINITE DIVISIBILITY OF SKS DISTRIBUTION

Following Shakil et al. [16], the pdf (1.3) of SKS distribution is given by

$$\frac{f'(x)}{f(x)} = \frac{a_{2p} x^{2p} + a_p x^p + a_0}{b_{p+1} x^{p+1}} \tag{4.1}$$

If we consider $H(x) = \ln f(x)$, then (4.1) can be written as

$$H'(x) = \frac{f'(x)}{f(x)} = \frac{a_{2p} x^{2p} + a_p x^p + a_0}{b_{p+1} x^{p+1}} .$$

Now, the density function $f(x)$ is log-concave, if $H(x) = \ln f(x)$ is concave. That is,

$$H''(x) < 0. \tag{4.2}$$

We have

$$H'(x) = \frac{a_{2p} x^{2p}}{b_{p+1} x^{p+1}} + \frac{a_p x^p}{b_{p+1} x^{p+1}} + \frac{a_0}{b_{p+1} x^{p+1}} = \left(\frac{a_{2p}}{b_{p+1}}\right) x^{p-1} + \left(\frac{a_p}{b_{p+1}}\right) x^{-1} + \left(\frac{a_0}{b_{p+1}}\right) x^{-p-1} .$$

If $\left(\frac{a_{2p}}{b_{p+1}}\right) = D_2$, $\left(\frac{a_p}{b_{p+1}}\right) = D_1$, $\left(\frac{a_0}{b_{p+1}}\right) = D_0$, we have

$H''(x) = D_2 x^{p-1} + D_1 x^{-1} + D_0 x^{-p-1}$, which on differentiating gives

$$H''(x) x^{p+2} = D_2 (p-1) x^{2p} - D_1 x^p - D_0 (p+1) . \tag{4.3}$$

For $x>0,\ x^{p+2}>0$, so, in view of equation (4.3), we have

$H''(x)<0$ only when $H''(x)x^{p+2}<0$. If $z(x)=H''(x)x^{p+2}$, then

$z(x)=D_2(p-1)x^{2p}-D_1x^p-D_0(p+1)<0$.

But $z(x)=H''(x)x^{p+2}$ is a quadratic function in x^p whose zeros are

$${x_1}^p=\frac{D_1}{2D_2(p-1)}-\frac{\sqrt{\Delta}}{2D_2(p-1)} \text{ and } {x_2}^p=\frac{D_1}{2D_2(p-1)}+\frac{\sqrt{\Delta}}{2D_2(p-1)}, \text{ where } \Delta={D_1}^2+4D_0D_2(p^2-1)$$

and $p\geq 2$. Now, the following Theorem easily follows:

Theorem 4.1. For the Pearson family with the form (4.1), we have

I. If $\frac{a_{2p}}{b_{p+1}}(p-1)>0$, $\Delta=\left(\frac{a_p}{b_{p+1}}\right)^2+4\left(\frac{a_0}{b_{p+1}}\right)\left(\frac{a_{2p}}{b_{p+1}}\right)(p^2-1)>0$, $\frac{a_p}{b_{p+1}}<0$ and $\left|\frac{a_p}{b_{p+1}}\right|\leq\sqrt{\Delta}$, for $0<x<\left(\frac{a_p}{2a_{2p}(p-1)}+\frac{b_{p+1}\sqrt{\Delta}}{2a_{2p}(p-1)}\right)^{\frac{1}{p}}$, the Pearson family is log-concave.

II. If $\frac{a_{2p}}{b_{p+1}}(p-1)>0$, $\Delta=\left(\frac{a_p}{b_{p+1}}\right)^2+4\left(\frac{a_0}{b_{p+1}}\right)\left(\frac{a_{2p}}{b_{p+1}}\right)(p^2-1)>0$, $\frac{a_p}{b_{p+1}}>\sqrt{\Delta}$, for $\left(\frac{a_p}{2a_{2p}(p-1)}-\frac{b_{p+1}\sqrt{\Delta}}{2a_{2p}(p-1)}\right)^{\frac{1}{p}}<x<\left(\frac{a_p}{2a_{2p}(p-1)}+\frac{b_{p+1}\sqrt{\Delta}}{2a_{2p}(p-1)}\right)^{\frac{1}{p}}$, the Pearson family is log-concave.

III. If $\frac{a_{2p}}{b_{p+1}}(p-1)<0$, $\Delta=\left(\frac{a_p}{b_{p+1}}\right)^2+4\left(\frac{a_0}{b_{p+1}}\right)\left(\frac{a_{2p}}{b_{p+1}}\right)(p^2-1)>0$, $\frac{a_p}{b_{p+1}}<0$ and $\left|\frac{a_p}{b_{p+1}}\right|>\sqrt{\Delta}$, for $0<x<\left(\frac{a_p}{2a_{2p}(p-1)}+\frac{b_{p+1}\sqrt{\Delta}}{2a_{2p}(p-1)}\right)^{\frac{1}{p}}$ or $x>\left(\frac{a_p}{2a_{2p}(p-1)}-\frac{b_{p+1}\sqrt{\Delta}}{2a_{2p}(p-1)}\right)^{\frac{1}{p}}$, the Pearson family is log-concave.

IV. If $\frac{a_{2p}}{b_{p+1}}(p-1)<0$, $\Delta=\left(\frac{a_p}{b_{p+1}}\right)^2+4\left(\frac{a_0}{b_{p+1}}\right)\left(\frac{a_{2p}}{b_{p+1}}\right)(p^2-1)=0$, and $0<x\neq\left(\frac{a_p}{2a_{2p}(p-1)}\right)^{\frac{1}{p}}$, the Pearson family is log-concave.

Now, in order to examine the logarithmic convexity and infinite divisibility of SKS distribution, we will need the following Lemmas.

Lemma 4.1 A function $f:R\to R_+$ is called log-concave on an open interval (a, b) if the function $\ln(f)$ is concave on (a, b), where $(a,b)\subset R$.

Lemma 4.2 (see Webster [19, page 207]; Boyd and Vandenberghe [2, page 104]): Let $f : \mathrm{I} \to R$ be a real-valued function on an open interval I of the real line R. Then f is log-convex if $f(x) > 0$ for all x in I and its logarithm, $\log f : \mathrm{I} \to R$, is convex.

Lemma 4.3 (see Steutel and Harn **[18, page 117]):** Let F be a distribution on R_+ with $F(0) = 0$, having a density f that is log-convex on $(0, \infty)$. Then F is infinitely divisible.

As noted in the introduction, the notions of log-convexity play major roles in various disciplines, such as economics, biology etc., for which the interested readers are referred to the references cited in the introduction.

Case I(a): From (1.4), we have

$$f(x) = C_1\, x^{\nu-1} e^{-\alpha x^p - \beta x^{-p}}$$

Taking log on both sides, we have

$$y = \ln f(x) = \ln C_1 + (\nu - 1)\ln x - \left(\alpha x^p + \beta x^{-p}\right)$$

Similarly, $y(x) = m + (\nu - 1)\ln x - \alpha\ x^p - \beta x^{-p}$, where $m = \ln C_1$

$$\frac{d\,y(x)}{d\,x} = \frac{\nu - 1}{x} - \alpha\, p\, x^{p-1} + \beta\, p\, x^{-p-1}$$

$$\frac{d^{\,2} y(x)}{d\,x^2} = -\frac{\nu - 1}{x^2} - \alpha\ p(p-1)\, x^{p-2} - \beta\ p(p+1) x^{-p-2}.$$

Obviously, $\frac{d^{\,2} y(x)}{d\,x^2} \ge 0$, provided $\frac{\nu - 1}{x^2} + \alpha\ p(p-1)\, x^{p-2} + \beta\, p(p+1) x^{-p-2} \le 0$. Multiplying both sides by x^{p+2}, we have, $\alpha\, p(p-1)\, x^{2p} + (\nu-1)\, x^p + \beta\, p(p+1) \le 0$, Consequently, by Lemma 4.2, we have $y = \ln\ f(x)$ is convex, which in turn implies $f(x)$ is log-convex (because $f(x)$, being a pdf is > 0). Hence $f(x)$ is infinitely divisible by lemma 4.3

Case I (b)

Now, if $x^p = u$, then above inequality reduces to $\alpha\, p(p-1) u^2 + (\nu - 1) u + \beta\, p\, (p+1) > 0$

If $\alpha\, p(p-1) > 0$, i.e., if $p > 1$, $\Delta_1 = (\nu-1)^2 - 4\alpha\, \beta\, p^2 (p^2 - 1) > 0$, then, by Lemma 5.1, the density function $f(x)$ is log-concave for x > 0 if $u_1 < x^p < u_2$, where $u_1 = \frac{-(\nu-1)}{2\alpha\, p(p-1)} - \frac{\sqrt{\Delta_1}}{2\alpha\, p(p-1)}$ and

$$u_2 = \frac{-(\nu-1)}{2\alpha\, p(p-1)} + \frac{\sqrt{\Delta_1}}{2\alpha\, p(p-1)}.$$

Case II: We have $\alpha > 0,\ \beta = 0,\ \nu > 0\ and\ p \in Z_+$

$$f(x) = C_2 x^{\nu-1} e^{-\alpha x^p} \qquad \text{where } C_2 = \frac{p\,\alpha^{\frac{\nu}{p}}}{\Gamma\left(\frac{\nu}{p}\right)} \tag{4.4}$$

and Z_+ is the set of all positive integers. We get:

$$\ln f(x) = \ln C_2 + (\nu - 1)\ln(x) - \alpha\ x^p$$

If we consider $y = \ln f(x)$, $m = \ln C_2$, we have

$$y = m + (\nu - 1)\ln(x) - \alpha\ x^p, \qquad (4.5)$$

from which we have

$$\frac{dy}{dx} = \frac{\nu - 1}{x} - \alpha\, p\, x^{p-1},$$

or

$$\frac{d^2 y}{dx^2} = -\frac{\nu - 1}{x^2} - \alpha\, p\,(p-1)x^{p-2},$$

$$\frac{d^2 y}{dx^2} < 0 \quad \text{provided} \quad \frac{\nu - 1}{x^2} + \alpha\, p\,(p-1)x^{p-2} > 0,$$

that is,

$$x \geq \left(\frac{1-\nu}{\alpha\, p\,(p-1)}\right)^{\frac{1}{p}} \quad \text{when the function is concave.}$$

Case III: When $\alpha = 0,\ \beta > 0, \nu < 0\, and\ \ p \in Z_+$

$$f(x) = C_3\, x^{\nu - 1} e^{-\beta x^{-p}}, \qquad C_3 = \frac{p\,\alpha^{\frac{\nu}{p}}}{\Gamma\left(\frac{\nu}{p}\right)}.$$

Now, taking natural log of the above pdf, we have

$$y = \ln f(x) = \ln C_3 + (\nu - 1)\ln x - \beta x^{-p}\ .$$

Let's consider $m = \ln C_3$. Then

$$y(x) = m + (\nu - 1)\ln x - \beta\ x^{-p},$$

or

$$\frac{d\,y(x)}{d\,x} = \frac{\nu - 1}{x} + \beta\ \ p\,x^{-p-1},$$

or

$$\frac{d^{\,2} y(x)}{d\,x^2} = -\frac{\nu - 1}{x^2} - \beta\, p\,(p+1)x^{-p-2},$$

or

$$\frac{d^{\,2} y(x)}{d\,x^2} < 0, \text{ provided } \frac{\nu - 1}{x^2} + \beta\, p\,(p+1)x^{-p-2} > 0.$$

That is,

$$x < \left(\frac{\beta\, m\, p\,(p+1)}{1-\nu}\right)^{\frac{1}{p}},$$

which is the required condition for probability density function to be concave in this case.

6. CONCLUDING REMARKS

The notions of the convexity and the logarithmic convexity play an extremely important role in mathematics, statistics, economics, political science, management science, information theory, and optimization. In mathematics, convexity and the logarithmic convexity of functions are used to prove various mathematical properties, particularly mathematical inequalities. Also, the logarithmic convexity of survival function $S(x)$ is used in reliability theory to determine the increase and decrease of failure rate. Jagadeesh and Chowdhry [10] has shown the important applications of the logarithmic concavity of reliability function in finance. Similarly, Flinn and Heckman [7] has used the logarithmic concavity of the function $R(x) = \int_x^{\infty} S(t)dt$, to analyze the optimal strategy for jobs and unemployment, and Bagnoli and Bergstrom [1] has used the log-concavity of the function $L(x) = \int_{-\infty}^{x} F(t)dt$ to develop a *marring market model*. Thus, it is expected that the results on the convexity and the logarithmic convexity of SKS distribution can be further explored that may find some important applications in statistics and some other areas listed above.

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