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Moment Properties of Generalized Order Statistics from Ailamujia Distribution

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Abstract: In this paper, we derive the explicit expression for the moments of generalized order statistics (*gos*) from Ailamujia distribution and some computational work is also carried out. Further, some recurrence relations for both single and product moments of *gos* for this distribution are derived and the results are deduced for order statistics and record values.

Keywords: Ailamujia distribution**,** generalized order statistics, order statistics, record values, recurrence relations.

1. Introduction

The concept of generalized order statistics (*gos*) was introduced by [1] and [2], which contains a variety of models of ascending ordered random variables, such as order statistics, upper record values, progressive type II censored order statistics, sequential order statistics and Pfeifer's records.

Let $X_1, X_2, ..., X_n$ be a sequence of independent identically distributed (*iid*) random variables with distribution function (*df*) F and probability density function (*pdf*) f. Let $k>0, n\in N, m\in \Re$ $\gamma_r = k+(n-r)(m+1)\geq M$ and $M_r = \sum_{j=r}^{n-1} m_j$. If the random variables $X(r,n,m,k), r=1,2,...n$ are r *gos* and have a joint *pdf* of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}(x_i)\right]^{m_i} f(x_i)\right)\left[\bar{F}(x_n)\right]^{k-1} f(x_n)\,, \tag{1}$$

on the cone $F^{-1}(0) < x_1 \leq .... \leq x_n < F^{-1}(1)$, then they are called *gos* of a sample with *df* $F(x)$.

Note that, if $m=0,\ k=1$ in (1), we get the joint *pdf* of order statistics and when $m=-1$ in (1), we get the joint *pdf* of the $k-th$ upper record values.

In view of (1), the marginal *pdf* of the $r-th$ *gos* $X(r,n,m,k)$ is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left(F(x)\right), \tag{2}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1\leq r<s\leq n$ is

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$$f_{X(r,n,m,k)X(s,n,m,k)}(x,y)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\left[\bar{F}(x)\right]^{m} f(x)g_m^{r-1}\left(F(x)\right)\left[h_mF(y)-h_mF(x)\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1} f(y)\,,$$

$$-\infty\le x<y\le\infty\,, \quad (3)$$

where,

$$\bar{F}(x)=1-F(x)\,,$$

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i\,,\quad r=1,2,\ldots,n\,,$$

$$h_m(x)=\begin{cases}-\frac{1}{m+1}(1-x)^{m+1} & m\ne -1\\ -\ln(1-x), & m=-1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(0)\,,\ x\in[0,1).$$

In reliability and supportability data analysis, the most widely used distributions are exponential, normal, logarithmic normal and Weibull distribution. In recent years, many new distributions are presented for various types of applications in engineering. Ailamujia (Эрланга) distribution proposed by [18], is one of the simplest and applicable distributions in direction of engineering applications. As alluded by [19] for the lifetime data problems one could consider Ailamujia distribution. Further, Fan [20] discussed the empirical Bayes estimation of the Ailamujia distribution, Li [21] discussed Bayesian test for lifetime performance index of Ailamujia distribution under squared error loss function.

A random variable X is said to have Ailamujia distribution if its *pdf* is of the form,

$$f(x)=4x\theta^2e^{-2\theta x};\quad x,\theta\ge 0\,, \quad (4)$$

with corresponding distribution function

$$F(x)=1-(1+2\theta x)e^{-2\theta x};\quad x,\theta\ge 0\,. \quad (5)$$

Now in view of (4) and (5), we have

$$4x\theta^2\bar{F}(x)=(1+2\theta x)f(x)\,. \quad (6)$$

The relation (6) will be utilized to establish recurrence relations for moments of *gos*.
Recurrence relations for moments of *gos* for some specific distributions are investigated by several authors in literature, for example, moment properties of generalized order statistics from exponential-Weibull lifetime distribution are discussed by [14], expectation identities based on recurrence relations of function of *gos* are given by [15]. For some additional results, one may refer to [16], [17] and more references therein.

In this paper, we study the moments of *gos* from Ailamujia distribution and derive the exact expression for single moments. We also establish some recurrence relations for the single and the product moments for this distribution. Further, various deductions and particular cases are discussed.

2. SINGLE MOMENTS

Theorem 2.1 For Ailamujia distribution as given in (1) for $1 \le r \le n$, $k \ge 1$ and $p = 1, 2, \ldots,$

$$E\left(X^p(r,n,m,k)\right) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} \sum_{v=0}^{\gamma_{r-u}-1} \binom{\gamma_{r-u}-1}{v} \binom{r-1}{u} (-1)^u \frac{(p+v+1)!}{(2\theta)^p \left[\gamma_{r-u}\right]^{p+v+2}}, \; m \ne -1 \quad (7)$$

$$E\left[X^p(r,n,-1,k)\right] = E[Y_r^{(k)}]^p = \frac{k^r}{(r-1)!} \sum_{c=0}^{\infty} \sum_{d=0}^{c+r-1} \sum_{e=0}^{d+k-1} \binom{d+k-1}{e} \binom{c+r-1}{d} a_c(r-1)(-1)^d \times \frac{(p+e+1)!}{(2\theta)^p (d+k)^{p+e+2}}, \quad m = -1 \quad (8)$$

Proof: From (2), we have

$$E\left(X^p(r,n,m,k)\right) = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^p \left[\bar{F}(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}\left(F(x)\right) dx. \quad (9)$$

On expanding $g_m^{r-1}\left(F(x)\right) = \left[\frac{1}{m+1}\left(1-\bar{F}(x)\right)^{m+1}\right]^{r-1}$ binomially in (9), we get

$$E\left(X^p(r,n,m,k)\right) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \int_0^\infty x^p \left[\bar{F}(x)\right]^{\gamma_r + (m+1)u - 1} f(x) dx \quad (10)$$

Using (4) and (5) in (10), we get

$$E\left(X^p(r,n,m,k)\right) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \int_0^\infty x^p \left[(1+2\theta x)e^{-2\theta x}\right]^{\gamma_{r-u}-1} 4x\theta^2 e^{-2\theta x} dx$$

$$E\left(X^p(r,n,m,k)\right) = \frac{4\theta^2 C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \int_0^\infty x^{p+1} (1+2\theta x)^{\gamma_{r-u}-1} e^{-2\gamma_{r-u}\theta x} dx$$

$$= \frac{(2\theta)^{2+v} C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} \sum_{v=0}^{\gamma_{r-u}-1} (-1)^u \binom{\gamma_{r-u}-1}{v} \binom{r-1}{u} \int_0^\infty x^{p+v+1} e^{-2\gamma_{r-u}\theta x} dx,$$

which upon simplification yields (7).

For $m = -1$, we have

$$E\left[X^p(r,n,-1,k)\right] = E\left[X^p_{U^{(k)}(r)}\right] = \frac{k^r}{(r-1)!} \int_0^\infty x^p \left[\bar{F}(x)\right]^{k-1} \left[-\ln\left(1-F(x)\right)\right]^{r-1} f(x) dx \quad (11)$$

where $E\left[X^p_{U^{(k)}(r)}\right]$ denotes the $p-th$ moment of $k-th$ upper record values

Using logarithmic expansion,

$$\left[-\ln(1-t)\right]^h = \left(\sum_{g=1}^{\infty} \frac{t^g}{g}\right)^h = \sum_{g=0}^{\infty} a_g(h) t^{g+h}, \qquad |t| < 1,$$

where $a_g(h)$ is the coefficient of t^{g+h} in the expansion of $\left(\sum_{g=1}^{\infty} \frac{t^g}{g}\right)^h$. (See [22]-[23]),

$$E\left[X^{p}_{U^{(k)}(r)}\right]=\frac{k^{r}}{(r-1)!}\sum_{c=0}^{\infty}a_{c}(r-1)\int_{0}^{\infty}x^{p}\left[\overline{F}(x)\right]^{k-1}\left[F(x)\right]^{c+r-1}f(x)dx.$$

Using (4) and (5) , we get

$$E\left[X^{p}_{U^{(k)}(r)}\right]=\frac{4\theta^{2}k^{r}}{(r-1)!}\sum_{c=0}^{\infty}\sum_{d=0}^{c+r-1}\binom{c+r-1}{d}a_{c}(r-1)\int_{0}^{\infty}x^{p+1}(1+2\theta x)^{d+k-1}e^{-2\theta xk}dx$$

$$E\left[X^{p}_{U^{(k)}(r)}\right]=\frac{k^{r}}{(r-1)!}\sum_{c=0}^{\infty}\sum_{d=0}^{c+r-1}\sum_{e=0}^{d+k-1}\binom{d+k-1}{e}\binom{c+r-1}{d}a_{c}(r-1)(-1)^{d}(2\theta)^{e}\int_{0}^{\infty}x^{p+e+1}e^{-2\theta x(d+k)}dx. \qquad (12)$$

Which after simplification yields (8).

Remark 2.1

a. When $m=0$, $k=1$ in (7), the explicit formula for the single moments of order statistics of Ailamujia distribution can be obtained as

$$E\left(X^{p}_{r:n}\right)=C_{r:n}\sum_{u=0}^{r-1}\sum_{v=0}^{n-r+u}\binom{n-r+u}{v}\binom{r-1}{u}(-1)^{u}\frac{(p+v+1)!}{(2\theta)^{p}\left[n-r+u+1\right]^{p+v+2}},$$

where, $C_{r:n}=\dfrac{n!}{(r-1)!(n-r)!}$.

b. When we put $k=1$ in (8), the explicit expression for the moments of upper record values for the Ailamujia distribution can be obtained as

$$E\left[X^{p}(r,n,-1,1)\right]=E\left[X^{p}_{U(r)}\right]=\frac{1}{(r-1)!}\sum_{c=0}^{\infty}\sum_{d=0}^{c+r-1}\sum_{e=0}^{d}\binom{d}{e}\binom{c+r-1}{d}a_{c}(r-1)(-1)^{d}\frac{(p+e+1)!}{(2\theta)^{p}(d+1)^{p+e+2}}.$$

In the following tables first four moments of order statistics and record values from Ailamujia distribution are computed by using results of Remark 2.1, for different arbitrary chosen parametric values of θ and for sample sizes $n=1,2,....,5.$

Table 2.1. First four moments of order statistics for Ailamujia distribution.

n	r	$\theta=5$				$\theta=10$			
		$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	1	0.2000000	0.0600000	0.0240000	0.0120000	0.1000000	0.0150000	0.0030000	0.0007500
2	1	0.1250000	0.0225000	0.0052500	0.0015000	0.0625000	0.0056200	0.0006500	0.0000900
	2	0.2750000	0.0975000	0.0427500	0.0225000	0.1375000	0.0243800	0.0053400	0.0014100
3	1	0.0962963	0.0130864	0.0022716	0.0004774	0.0481481	0.0032716	0.0002839	0.0000298
	2	0.1824070	0.0413272	0.0112068	0.0035453	0.0912037	0.0103318	0.0014009	0.0002219
	3	0.3212960	0.1255860	0.0585216	0.0319774	0.1606480	0.0313966	0.0073152	0.0019986
4	1	0.0804688	0.0090234	0.0012803	0.0002183	0.0402344	0.0022559	0.0001600	0.0000136
	2	0.1437790	0.0252754	0.0052456	0.0012547	0.0718895	0.0063188	0.0006557	0.0000784
	3	0.2210360	0.0573790	0.0171680	0.0058359	0.1105180	0.0143447	0.0021460	0.0003647
	4	0.3547160	0.1483220	0.0723061	0.0406912	0.1773580	0.0370806	0.0090380	0.0025432
5	1	0.0702080	0.0068083	0.0008297	0.0001208	0.0351040	0.0017021	0.0001037	0.0000076
	2	0.1215120	0.0178839	0.0030824	0.0006079	0.0607559	0.0044709	0.0003853	0.0000379
	3	0.1771800	0.0363626	0.0849043	0.0022248	0.0885899	0.0090906	0.0010613	0.0001391

n	r	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
	4	0.2502730	0.0713899	0.0229530	0.0082432	0.1251370	0.0178475	0.0028691	0.0005152
	5	0.380827	0.1675550	0.0846444	0.0488032	0.1904140	0.0418888	0.0105806	0.0030500
n	r	$\theta=15$				$\theta=20$			
		$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	1	0.0666778	0.0066788	0.0008889	0.0001481	0.0500000	0.0037500	0.0003700	0.0000468
2	1	0.0416667	0.0025000	0.0001944	0.0001851	0.0312500	0.0014063	0.0000820	0.0000058
	2	0.0916667	0.0108333	0.0015833	0.0002778	0.0687500	0.0060938	0.0006679	0.0000878
3	1	0.0320989	0.0014541	0.0000841	0.0000059	0.0240741	0.0008179	0.0000355	0.0000186
	2	0.0608025	0.0045919	0.0004150	0.0000437	0.0456019	0.0025829	0.0001751	0.0000138
	3	0.1070900	0.0139500	0.0021675	0.0003947	0.0803241	0.0078491	0.0009144	0.0001249
4	1	0.0268229	0.0010026	0.0000474	0.0000026	0.0201172	0.0005639	0.0000200	0.0000008
	2	0.0479263	0.0028084	0.0001942	0.0000154	0.0359400	0.0015797	0.0000819	0.0000049
	3	0.0736786	0.0063754	0.0006350	0.0000720	0.0552529	0.0035862	0.0002682	0.0000227
	4	0.1182390	0.0164802	0.0026780	0.0005023	0.0886791	0.0092701	0.0011297	0.0001689
5	1	0.0234027	0.0007565	0.0000307	0.0000014	0.0175520	0.0004255	0.0000129	0.0000005
	2	0.0405039	0.0019871	0.0001142	0.0000075	0.0303779	0.0011177	0.0000482	0.0000023
	3	0.0590599	0.0040428	0.0003145	0.0000274	0.0442949	0.0022727	0.0001327	0.0000086
	4	0.0834244	0.0079322	0.0008502	0.0001018	0.0625683	0.0044618	0.0003641	0.0000322
	5	0.1269420	0.0186273	0.0031349	0.0006025	0.0952068	0.0104722	0.0013226	0.0001907

In Table 2.1, it may be noted that the well known property of order statistics $E\left(\sum_{i=1}^{n} X_{i:n}^{p}\right) = nE(X)^{p}$ (David and Nagaraja [12]) is satisfied.

Table 2.2. First four moments of record statistics for Ailamujia distribution.

r	$\theta=5$				$\theta=10$			
	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	0.2000000	0.060000	0.024000	0.0120000	0.1000000	0.0150000	0.0030000	0.0007500
2	0.3400000	0.1480000	0.0786001	0.0494020	0.1701811	0.0370201	0.0098380	0.0030901
3	0.4669511	0.2614651	0.1710731	0.1282651	0.2334710	0.0653641	0.0213810	0.0080200
4	0.5871001	0.3988811	0.3083511	0.2675601	0.2935321	0.0997111	0.0385451	0.0167141
5	0.7034200	0.5595650	0.4973333	0.4888880	0.3517050	0.1398793	0.0621592	0.0305480
	$\theta=15$				$\theta=20$			
1	0.0666667	0.0066652	0.0008810	0.0001480	0.0500000	0.0037462	0.0003750	0.0000041
2	0.1134454	0.0164500	0.0029101	0.0006050	0.0850789	0.0092453	0.0012253	0.0001851
3	0.1556500	0.0290402	0.0063411	0.0015751	0.1167350	0.0163400	0.0026701	0.0005100
4	0.1957000	0.0443160	0.0114222	0.0033072	0.1467751	0.0249257	0.0048190	0.0010511
5	0.2344711	0.0621666	0.0184161	0.0060431	0.1758501	0.0349666	0.0077784	0.0019000

Theorem 2.2. For the distribution as given in (4) with $n \in N$, $m \in \Re$, $2 \le r \le n$, $p = 2,3...$,

$$E\left(X^{p}(r,n,m,k)\right) - E\left(X^{p}(r-1,n,m,k)\right) = \frac{p}{4\theta^{2}\gamma_{r}}\left[E\left(X^{p-2}(r,n,m,k)\right) + 2\theta E\left(X^{p-1}(r,n,m,k)\right)\right], \quad (13)$$

$$E\left(X^{p}(r,n,m,k)\right)-E\left(X^{p}(r-1,n-1,m,k)\right) = \frac{p}{4\theta^{2}\gamma_{1}}\left[E\left(X^{p-2}(r,n,m,k)\right)+2\theta E\left(X^{p-1}(r,n,m,k)\right)\right]. \tag{14}$$

Proof: From Athar and Islam [24], we have

$$E\left[\xi\left\{X(r,n,m,k)\right\}\right]-E\left[\xi\left\{X(r-1,n,m,k)\right\}\right]=\frac{C_{r-2}}{(r-1)!}\int_{-\infty}^{\infty}\xi'(x)\left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}(F(x))dx.$$

Let $\xi(x)=x^{p}$, then we have

$$E\left(X^{p}(r,n,m,k)\right)-E\left(X^{p}(r-1,n,m,k)\right)=\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}(F(x))dx \tag{15}$$

Now, using (6) in (15), we obtain

$$E\left(X^{p}(r,n,m,k)\right)-E\left(X^{p}(r-1,n,m,k)\right)=\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{0}^{\infty}x^{p-2}\left[\bar{F}(x)\right]^{\gamma_r-1} g_m^{r-1}(F(x))f(x)\left[\frac{(1+2\theta x)}{4\theta^{2}}\right]dx$$

$$E\left(X^{p}(r,n,m,k)\right)-E\left(X^{p}(r-1,n,m,k)\right)=\frac{pC_{r-1}}{4\theta^{2}\gamma_r(r-1)!}\int_{0}^{\infty}x^{p-2}\left[\bar{F}(x)\right]^{\gamma_r-1} g_m^{r-1}(F(x))f(x)(1+2\theta x)dx$$

$$\begin{aligned}E\left(X^{p}(r,n,m,k)\right)-E\left(X^{p}(r-1,n,m,k)\right)&=\frac{pC_{r-1}}{4\theta^{2}\gamma_r(r-1)!}\int_{0}^{\infty}x^{p-2}\left[\bar{F}(x)\right]^{\gamma_r-1} g_m^{r-1}(F(x))f(x)dx\\&\quad+\frac{pC_{r-1}}{2\theta\gamma_r(r-1)!}\int_{0}^{\infty}x^{p-1}\left[\bar{F}(x)\right]^{\gamma_r-1} g_m^{r-1}(F(x))f(x)dx\end{aligned} \tag{16}$$

After simplifying (16), we get (13).

Now consider the following identity from Corollary 1.1 of [1] (pp-116), we get

$$\left[k+(n-r-1)(m+1)\right]E\left(X^{p}(r,n,m,k)\right)+r(m+1)E\left(X^{p}(r+1,n,m,k)\right) = \left[k+(n-1)(m+1)\right]E\left(X^{p}(r,n-1,m,k)\right) \tag{17}$$

By replacing r to $r-1$ in (17), we get

$$\gamma_r E\left[X^{p}(r-1,n,m,k)\right]+(r-1)(m+1)E\left[X^{p}(r,n,m,k)\right]=\gamma_1 E\left[X^{p}(r-1,n-1,m,k)\right] \tag{18}$$

Now, using (18) in (13), we get

$$\frac{\gamma_1}{\gamma_r}\left[E\left(X^{p}(r,n,m,k)\right)-E\left(X^{p}(r-1,n-1,m,k)\right)\right] = \frac{p}{2\theta\gamma_r}\left[\frac{1}{2\theta}E\left(X^{p-2}(r,n,m,k)\right)+E\left(X^{p-1}(r,n,m,k)\right)\right] \tag{19}$$

After rearranging the terms of (19), we get (14).

Remark 2.2. Substituting $m=0, k=1$ in (13), we deduce recurrence relation for single moments of order statistics from Ailamujia distribution.

$$E\left(X_{r:n}^{p}\right)-E\left(X_{r-1:n}^{p}\right)=\frac{p}{4\theta^{2}(n-r+1)}\left[E\left(X_{r:n}^{p-2}\right)+2\theta E\left(X_{r:n}^{p-1}\right)\right].$$

Remark 2.3. $m=-1, k=1$ in (13), we get the result for single moments of upper record values fromAilamujia distribution is deduced as

$$E\left(X_{U(r)}^{p}\right)-E\left(X_{U(r-1)}^{p}\right)=\frac{p}{4\theta^2}\left[E\left(X_{U(r)}^{p-2}\right)+2\theta E\left(X_{U(r)}^{p-1}\right)\right].$$

4. PRODUCT MOMENTS

Theorem 3.1. For the distribution as given in (4) with $n\in N$, $m\in\Re, 1\le r\le s\le n$, $p=1,2,\ldots,\ q\ge 2$.

$$E\left[X^p(r,n,m,k)X^q(s,n,m,k)\right]-E\left[X^p(r,n,m,k)X^q(s-1,n,m,k)\right]$$
$$=\frac{q}{4\theta^2\gamma_s}\left[E\left(X^p(r,n,m,k)X^{q-2}(s,n,m,k)\right)+2\theta E\left(X^p(r,n,m,k)X^{q-1}(s,n,m,k)\right)\right]. \quad (20)$$

Proof: In view of Athar and Islam [24], we get

$$E\left[\xi\left(X(r,n,m,k)\right)\xi\left(X(s,n,m,k)\right)\right]-E\left[\xi\left(X(r,n,m,k)\right)\xi\left(X(s-1,n,m,k)\right)\right]$$
$$=\frac{C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x}^{\infty}\frac{\partial}{\partial y}\xi(x,y)\left[\bar F(x)\right]^m f(x)g_m^{r-1}\left(F(x)\right)\left[h_mF(y)-h_mF(x)\right]^{s-r-1}\left[\bar F(y)\right]^{\gamma_s}dydx. \quad (21)$$

Let $\xi(x,y)=\xi_1(x).\xi_2(y)=x^py^q$.

On using relation (6) in (21), we get

$$E\left[X^p(r,n,m,k)X^q(s,n,m,k)\right]-E\left[X^p(r,n,m,k)X^q(s-1,n,m,k)\right]=$$
$$\frac{qC_{s-1}}{4\theta^2\gamma_s(r-1)!(s-r-1)!}\int_0^{\infty}\int_x^{\infty}x^py^{q-2}\left[\bar F(x)\right]^m f(x)g_m^{r-1}\left(F(x)\right)\left[h_mF(y)-h_mF(x)\right]^{s-r-1}\left[\bar F(y)\right]^{\gamma_s-1}f(y)dydx$$
$$+\frac{qC_{s-1}}{2\theta\gamma_s(r-1)!(s-r-1)!}\int_0^{\infty}\int_x^{\infty}x^py^{q-1}\left[\bar F(x)\right]^m f(x)g_m^{r-1}\left(F(x)\right)\left[h_mF(y)-h_mF(x)\right]^{s-r-1}\left[\bar F(y)\right]^{\gamma_s-1}f(y)dydx,$$

after simplification, we get the result of (20).

Remark 3.1. Putting $m=0$, $k=1$ in (20), we get the recurrence relation for the product of order statistics from Ailamujia distribution

$$E\left(X_{r:n}^pX_{s:n}^q\right)-E\left(X_{r:n}^pX_{s-1:n}^q\right)=\frac{q}{4\theta^2(n-s+1)}\left[E\left(X_{r:n}^pX_{s:n}^{q-2}\right)+2\theta E\left(X_{r:n}^pX_{s:n}^{q-1}\right)\right].$$

Remark 3.2. Setting $m=-1$, $k=1$ in (20), we get the relation for the product moments of upper record values from Ailamujia distributions

$$E\left(X^p(r,n,-1,1)X^q(r,n,-1,1)\right)-E\left(X^p(r,n,-1,1)\left(X^q(s-1,n,-1,1)\right)\right)=$$
$$\frac{p}{4\theta^2}\left[E\left\{\left(X^p(r,n,-1,1)\right)\left(X^{q-2}(s,n,-1,1)\right)\right\}+2\theta E\left\{\left(X^p(r,n,-1,1)\right)\left(X^{q-1}(s,n,-1,1)\right)\right\}\right].$$

Remark 3.3. At $p=0$ in equation (20), we get the result as established in (13).

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REFERENCES

[1] U. Kamps, "A Concept of Generalized Order Statistics", B.G. Teubner Stuttgart, Germany, 1995.

[2] U. Kamps, "A concept of generalized order statistics" pp. 1-23, 1995.

[3] U. Kamps and U. Gather, "Characteristic property of generalized order statistics for exponential distribution,"*Applicationes Mathematicae* (Warsaw), vol. 24, no. 4, pp. 383-391, 1997.

[4] C. Keseling, "Conditional distributions of generalized order statistics and some characterizations," *Metrika*, vol.49, no. 1, pp. 27-40, 1999.

[5] E. Cramer, and U. Kamps, "Relations for expectations of functions of generalized order statistics," *Journal of Statistical Planning and Inference*, vol. 89, no. 1-2, pp. 79-89, 2000.

[6] M. Ahsanullah, "Generalized order statistics from exponential distribution," *Journal of Statistical Planning andInference*, vol. 85, no. 1-2, pp. 85-91, 2000.

[7] M. Habibullah and M. Ahsanullah, "Estimation of parameters of a Pareto distribution by generalized ordestatistics," *Communications in Statistics—Theory and Methods*, vol. 29, no.7, pp. 1597-1609, 2000.

[8] M.Z. Raqab, "Optimal prediction-intervals for the exponential distribution based on generalized order statistics," *IEEE Transactions on Reliability*, vol. 50, no. 1, pp. 112-115, 2001.

[9] U. Kamps and E. Cramer, "On distributions of generalized order statistics," *Statistics*, vol. 35, no. 3, pp. 269- 280, 2001.

[10] A.A. Ahmad and A.M. Fawzy, "Recurrence relations for single moments of generalized order statistics from doubly truncated distribution," *Journal of Statistical Planning and Inference*, vol. 117, no. 2, pp. 241–249, 2003.

[11] C. Keseling, "Conditional distributions of generalized order statistics and some characterizations," *Metrika*, vol. 49, no. 1, pp. 27-40, 1999.

[12] H.A. David and H.N. Nagaraja, *Order Statistics*, 3rd edition, John Wiley, New York, 2003.

[13] D. Kumar, Generalized order statistics from Kumaraswamy distribution and its characterization. *Tamsui Oxford journal of Mathematical Sciences*, *27*, pp.463-476, 2011.

[14] R.U. Khan, and M. A. Khan. "Moment properties of generalized order statistics from exponential-Weibull lifetime distribution." *Journal of Advanced Statistics* vol. 1, no. 3, 147, 2016.

[15] Z. Anwar, H. Athar, and R.U. Khan, "Expectation identities based on recurrence relations of function of generalized order statistics" *Journal of Statistical Research*, vol. 41, no. 2, pp.93-102, 2007.

[16] Nayabuddin and H. Athar, "Recurrence relations for single and product moments of generalized order statistics from Marshall–Olkin extended Pareto distributions" *Communications in Statistics-Theory and Methods*, vol. 46 ,no. 16, pp.7820-7826, 2017.

[17] N. Gupta, and Z. Anwar, "Relations for Single and Product Moments of Odds Generalized Exponential-Pareto Distribution Based on Generalized Order Statistics and its Characterization", *Statistics, Optimization & Information Computing*, (to appear).

[18] H. Q. Lv., L. H . Gao, C. L. Chen, "Эрланга distribution and its application in supportability data analysis", *Journal of Academy of Armored Force Engineering*, vol. 16, no. 3, pp. 48-52, 2002.

[19] H. Zhou, "Data Processing and Technology Application in Bayes and Empirical Bayes Reliability Analysis of Parameter of Ailamujia Distribution" *Advanced Materials Research*, vol. 951,pp. 249-252. Trans Tech Publications, 2014.

[20] G.Fan,"Asymptotic Optimal Empirical Bayes Estimation of the Parameter of ЭРланга Distribution", *American Journal of Applied Mathematics*, vol. 4,no. 6, pp.283-288, 2016.

[21] L. Lanping, "Bayesian Test for Lifetime Performance Index of Ailamujia Distribution Under Squared Error Loss Function", *Pure and Applied Mathematics Journal*, vol. 5, no. 6, pp. 181-185, 2016.

[22] N. Balakrishnan and A. C. Cohan,"Order Statistics and Inference: Estimation Methods" , *Elsevier, Academic Press, San Diego*, 1991.

[23] A. I. Shawky, and R. A . Bakoban, "Certain characterizations of the exponentiated gamma distribution", *JATA*, 2008.

[24] H. Athar, and H.M. Islam, "Recurrence relations for single and product moments of generalized order statistics from a general class of distribution", *METRON*, vol 62, no.3, pp. 327-337, 2004.

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