Orbit Determination Method Using Thorne's Series Solution of Lambert Problem

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ABSTRACT

In the present paper, use will be made of Thorne's series solution of Lambert problem to develop an orbit determination method valid for hyperbolic and elliptical orbits. Numerical illustrations are given.

KEYWORDS: Orbit determination, Lambert problem, Space dynamics.

INTRODUCTION

Lambert problem of space researches is concerned with the determination of an orbit from two position vectors and the time of flight (Danby 1988). It has very important applications in the areas of rendezvous, targeting, guidance (Noton 1998) and interplanetary missions (Eagle 1991).

Solutions to Lambert's problem abound in the literature, as they did even in Lambert's time shortly after his original formulation in 1716. Examples are Lambert's original geometric formulation, which provides equations to determine the minimum-energy orbit, and the original Gaussian formulation, which gives geometrical insight into the problem.

Up to the year 1965, a fairly comprehensive list of references on Lambert's problem are given in (Escobal 1965), (Herrick 1971) and (Battin1964). In 1969, (Lancaster and Blanchard 1969)- also (Mansfield 1989)- established unified forms of Lambert's problem, and in 1990 (Gooding, 1990) developed a procedure for the solution. An algorithm for the universal Lambert's problem based on iterative scheme that could be made converge for all coin motion was established by (Sharaf, et al. 2003).

Each of the above methods is characterized primarily by: (1) a particular form of the time of flight equation and, (2) a particular independent variable to be used in an iteration algorithm to determine the orbital elements. For examples the semi-major axis a, and in some cases, the orbital parameter p is also used (Sun, 1981). The main disadvantage of using p as an independent variable is that a singularity occurs in the equation for collinear position vector. The independent variables used in Lambert's problem, all satisfied transcendental equation. Transcendental equations are usually solved by iterative methods, which in turn need: (a) initial guess, (b) an iterative scheme. In fact, these two points are not separated from each other, but there is a full agreement that even accurate iterative schemes are extremely sensitive to the initial guess. Moreover, in many cases the initial guess may lead to drastic situation between divergent and very slow convergent solutions.

To avoid these difficulties, analytical tool is to be used .In this respect (Thorne and Bain, 1995) developed an important and useful series solution to the Lambert problem. It is equally valid for hyperbolic and elliptical transfer times that are less than the time for minimum-energy transfer.

In the present paper use will be made of the above analytical solution and the basic Lambert's parameters to develop an orbit determination method valid for hyperbolic and elliptical orbits. Numerical illustrations are also given.

BASIC FORMULATIONS

LAGRANGE'S FUNCTIONS

On any of the two bodies orbits (elliptic, parabolic, or hyperbolic) we have:

$$\mathbf{r}_2 = F\,\mathbf{r}_1 + G\,\mathbf{v}_1, \tag{1}$$

$$\mathbf{v}_2 = \dot{F}\,\mathbf{r}_1 + \dot{G}\,\mathbf{v}_1\ , \tag{2}$$

where $(\mathbf{r}_1, \mathbf{v}_1)$are the position and velocity vectors at time t_1 while $(\mathbf{r}_2, \mathbf{v}_2)$ are the corresponding vectors at another time t_2. The coefficients F and G are functions of $\Delta t = t_2 - t_1$ and known as the *Lagrange* F and G functions, $\dot{F}$ and $\dot{G}$ are their time derivatives. $\mathbf{r}$ and $\mathbf{v}$ are given in components as:

$$\mathbf{r} = \mathbf{i}_x x + \mathbf{i}_y y + \mathbf{i}_z z \tag{3}$$

$$\mathbf{v} = \mathbf{i}_x \dot{x} + \mathbf{i}_y \dot{y} + \mathbf{i}_z \dot{z}\ , \tag{4}$$

$\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are the unit vectors along the coordinate axes x, y and z respectively, and

$$r = (x^2 + y^2 + z^2)^{1/2} \tag{5}$$

There are different forms of Lagrange's functions. Here we use only the expression of the G function in terms of the diffrence $\theta = f_2 - f_1$ in the true anomalies (θ usually called the *transfer angle*):

$$G = \frac{r_1 r_2}{h}\sin\theta \tag{6}$$

where h is the magnitude of the angular momentum vector $\mathbf{h}$,

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} \tag{7}$$

BASIC EQUATIONS OF LAMBERT'S PROBLEM

The basic equations of Lambert's problem are given in many references and may be summerized as follows:

- Traveling between two specified points can take the long way or the short way. For the long way, θ exceeds 180°, while for the short way θ<180°, so that

  $$\cos\theta = \frac{\langle \mathbf{r}_1, \mathbf{r}_2 \rangle}{r_1 r_2}; \quad \sin\theta = \frac{|\,\mathbf{r}_1 \times \mathbf{r}_2\,|}{r_1 r_2} \quad \text{or} \quad \sin\theta = t_m\sqrt{1 - \cos^2\theta} \tag{8}$$

  where t_m is (+1) for short way transfers and (-1) for long way transfers.

- The length of the chord between the two position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is:

  $$c = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2\cos\theta} \tag{9}$$

- Define the semiperimeter, s, as half the sum of the sides of the triangle created by the position vectors and the chord, then

$$s = \frac{1}{2}(r_1 + r_2 + c) \quad (10)$$

Some constants were introduced for elliptic and hyperbolic orbits, of these are (α_e, β_e) and (α_h, β_h), where

$$\sin(\frac{1}{2}\alpha_e) = \sqrt{\frac{s}{2a}}, \qquad \sin(\frac{1}{2}\beta_e) = \sqrt{\frac{s-c}{2a}}, \quad (11)$$

$$\sinh(\frac{1}{2}\alpha_h) = \sqrt{\frac{s}{-2a}}, \qquad \sinh(\frac{1}{2}\beta_h) = \sqrt{\frac{s-c}{-2a}}. \quad (12)$$

and a is the semi-major axis of the orbit

- The constants α's and β's are related to the eccentric and hyperbolic anomalies by:

$$\Delta E = E_2 - E_1 = \alpha_e - \beta_e \quad (13)$$

$$\Delta H = H_2 - H_1 = \alpha_h - \beta_h \quad (14)$$

- Finally (Battin,1999), the general time of flight, some times called Lambert's equation, is, given for elliptic and hyperbolic orbits as:

$$t_e = \sqrt{\frac{a^3}{\mu}}[(\alpha_e - \sin\alpha_e) - (\beta_e - \sin\beta_e)] \quad (15)$$

$$t_h = \sqrt{\frac{-a^3}{\mu}}[(\sinh\alpha_h - \alpha_h) - (\sinh\beta_h - \beta_h)] \quad (16)$$

where μ is the gravitational parameter.

THORNE'S SOLUTION OF LAMBERT PROBLEM

Let us frist define the quantity T:

$$T = t/t_p - 1 \quad (17)$$

As nondimentional time parameter, where t = t_e or t_h is the desired flight (Equation (15) or (16)) and t_p is the known parabolic flight time between the two given position vectors,

$$t_p = \frac{2}{3}\sqrt{\frac{s^3}{\mu}}\left\{1 - \left(\frac{s-c}{s}\right)^{3/2}\right\}. \quad (18)$$

Thorne and Bain (1995) established the following power series for T in the unknown semi-major axis a as:

$$T=\sum_{i=1}^{\infty}\frac{\left(1-\left(\frac{s-c}{s}\right)^{i+\frac{3}{2}}\right)}{\left(1-\left(\frac{s-c}{s}\right)^{\frac{3}{2}}\right)}\frac{\left(\frac{1}{2}\right)_i\left(\frac{3}{2}\right)_i}{\left(\frac{5}{2}\right)_i i!}\left(\frac{s}{2a}\right)^i=\sum_{i=1}^{\infty}A_i\left(\frac{s}{2a}\right)^i \qquad (19)$$

The symbols $(x)_i$ are Pochhammer symbols, defined by $(x)_i = x(x+1)...(x+i-1)$.

Having caculated the A's terms to the desired order from Equation (19), we can use them to find the B's coefficients of the following series:

$$a=\left(\frac{s}{2}\right)\sum_{i=1}^{\infty}B_i T^{i-2} \qquad (20)$$

This sereies solution is to be used only when the time parametre lies in the interval $(-1 < T < 1)$.

If T is exactly zero, the path is parabolic, and the orbit geometry is already known. If $T \prec 0.0$, the transefer is hyperbolic, and if $T \succ 0.0$, it is elliptic.

To compute the B's coefficients in Equation (20) two steps to be performed:

First construct the lower triangular matrix, $\boldsymbol{Q}[q_{i,j}]$ from the following recursive formulae:

$$q_{1,1}=A_1^{-1}$$

$$q_{i,j}=\sum_{k=1}^{i-1}q_{i-k,j-1}\,q_{k,1} \qquad (i=2,3,4,\cdots),1\prec j\le i \qquad (21)$$

$$q_{i,1}=\sum_{k=1}^{i-1}\left(\frac{-1}{A_1}\right)q_{i,k+1}\,A_{k+1}$$

The second step, is to to apply the matrix $\boldsymbol{Q}$ on the known series coefficients A_i to obtain the final series coefficients B_i as:

$$B_i=\sum_{k=1}^{i-1}q_{i-1,k}A_{k+1} \qquad (22)$$

Finally, the semi-major axis, a is, given by Equation (20), generally, the closer T is to zero, the more accurate a will be.

ORBITAL DETERMIATION METHOD

The radial distance for elliptic and hyperbolic orbits is given respectively as:

$$r=a(1-e\cos E) \qquad (23)$$

and

$$r=a(\cosh H-1) \qquad (24)$$

where e is the orbital eccentricity . Since $a \prec 0$, for hyperbolic orbits, it follows from Equations (13), (14), (23) and (24) that

$$ea\cos E_1=a-r_1 \qquad (25.1)$$

$$ea\sin E_1 = \frac{1}{\sin(\alpha_e - \beta_e)}\left((r_2 - a) - (r_1 - a)\cos(\alpha_e - \beta_e)\right) \tag{25.2}$$

$$ea\cosh H_1 = r_1 - a \tag{26.1}$$

$$ea\sinh H_1 = \frac{1}{\sinh(\alpha_h - \beta_h)}\left((r_2 - a) - (r_1 - a)\cosh(\alpha_h - \beta_h)\right) \tag{26.2}$$

Also we have:

$$\frac{r_1}{a}\cos f_1 = \cos E_1 - e \tag{27.1}$$

$$\frac{r_1}{a}\sin f_1 = (1-e^2)^{1/2}\sin E_1 \tag{27.2}$$

$$\frac{r_1}{a}\cos f_1 = e - \cosh H_1 \tag{28.1}$$

$$\frac{r_1}{a}\sin f_1 = (e^2-1)^{1/2}\sinh H_1 \tag{28.2}$$

For both orbits, the components (h_x, h_y, h_z) of the angular momentum vector $\boldsymbol{h}$ are related to the velocity components $(\dot{x}, \dot{y}, \dot{z})$ and the orbital elements by:

$$h_x = y\dot{z} - z\dot{y} = h\sin\Omega\sin i\,, \tag{29.1}$$

$$h_y = z\dot{x} - x\dot{z} = -h\cos\Omega\sin i \tag{29.2}$$

$$h_z = x\dot{y} - y\dot{x} = h\cos i \tag{29.3}$$

where i is the orbital inclination and Ω the longitude of the accending node.

From the second Equation in (8), we have

$$r_1 r_2 \sin\theta = (\lambda^2 + \chi^2 + \nu^2)^{1/2} \tag{30}$$

where

$$\lambda = y_1 z_2 - z_1 y_2 \quad ; \quad \chi = z_1 x_2 - x_1 z_2 \quad ; \quad \nu = x_1 y_2 - y_1 x_2 . \tag{31}$$

Using Equations (1), (6) and (27) we get:

$$\frac{\lambda}{(\lambda^2 + \chi^2 + \nu^2)^{1/2}} = \sin\Omega\sin i \quad ; \tag{32.1}$$

$$\frac{\chi}{(\lambda^2 + \chi^2 + \nu^2)^{1/2}} = -\cos\Omega\sin i \ ; \tag{32.2}$$

$$\frac{\nu}{(\lambda^2 + \chi^2 + \nu^2)^{1/2}} = \cos\Omega. \tag{32.3}$$

Finally, for both orbits, we have:

$$r_1\cos u = x_1\cos\Omega + y_1\sin\Omega \tag{33.1}$$

$$r_1\sin u = (-x_1\sin\Omega + y_1\cos\Omega)/\cos i \tag{33.2}$$

where u is the argument of latitude given in terms of the longitude of pericenter ω and the true anomaly f_1 by:

$$u = \omega + f_1 \quad (34)$$

COMPUTATIOAL DEVELOPMENTS

COMPUTATIOAL ALGORIHM

- **Purpose:** To detrmine the orbital element for elliptic or hyberbolic orbite using Thorne's seres soution of Lambert problem
- **Input:** $x_1, y_1, z_1; x_2, y_2, z_2; \mu, n$ (order), t (time)
- **Output:** $a, e, i, \omega, \Omega, u$
- **Computational sequence**
  1. $r_1 = (x_1^2 + y_1^2 + z_1^2)^{1/2}$,
  2. $r_2 = (x_2^2 + y_2^2 + z_2^2)^{1/2}$,
  3. $\mathrm{co} = (x_1 x_2 + y_1 y_2 + z_1 z_2) / r_1 r_2$,
  4. $c = (r_1^2 + r_2^2 - 2 r_1 r_2 \times \mathrm{co})^{1/2}$,
  5. $s = 0.5(r_1 + r_2 + c)$,
  6. $k = (s - c)/s$
  7. $t_p = \frac{2}{3}\sqrt{\frac{s^3}{2\mu}}\,(1 - k^{1.5})$,
  8. $T = t / t_p - 1$,
  9. $A_i = \frac{(0.5)_i (1.5)_i (1 - k^{i+1.5})}{i!(1 - k^{1.5})(2.5)_i}$ **Error! Objects cannot be created from editing field codes.**
  10. $q_{i,j}$ from Equation (21) ,
  11. B_i ;**Error! Objects cannot be created from editing field codes.** from Equation (22),
  12. a from Equation (20),
  13. If $T \succ 0$, then
      - α and β from Equation (11)
      - $Q_1 = (r_2 - a - (r_1 - a)\cos(\alpha - \beta)) / \sin(\alpha - \beta)$
      - $Q_2 = a - r_1$
      - $E_1 = \tan^{-1}(Q_1 / Q_2)$
      - $e = (Q_1^2 + Q_2^2)^{1/2} / a$
      - $f_1 = \tan^{-1}\left(\frac{\sin E_1 \sqrt{1 - e^2}}{\cos E_1 - e}\right)$

      Else
      - α and β from Equation (12)
      - $Q_1 = (r_2 - a - (r_1 - a)\cosh(\alpha - \beta)) / \sinh(\alpha - \beta)$
      - $Q_2 = -a + r_1$
      - $E_1 = \tanh^{-1}(Q_1 / Q_2)$

- $e = -(Q_1^2 + Q_2^2)^{1/2} / a$
- $f_1 = \tan^{-1}\left(\dfrac{\sinh E_1 \sqrt{1-e^2}}{\cosh E_1 - e}\right)$

And If

14. λ, χ and ν from Equation (31),
15. $\Omega = \tan^{-1}(\lambda/(-\chi))$,
16. $i = \tan^{-1}\left(\sqrt{\dfrac{\lambda^2+\chi^2}{\nu^2}}\right)$,
17. $u = \tan^{-1}\left(\dfrac{y_1 \cos\Omega - x_1 \sin\Omega}{\cos i\,(x_1 \cos\Omega + y_1 \sin\Omega)}\right)$,
18. $\omega = u - f_1$
19. End

NUMERICAL APPLICATIONS

1. Sputnik III (Sconzo,1962)

$t_1 = 1959$, May 14.044194 UT

$t_2 = 1959$, May 14.049333 UT

$t = 444.01$ s

$x_1 = -1597.82$ km	$x_2 = 145.779$ km
$y_1 = -3706.07$ km	$y_2 = -5734.34$ km
$z_1 = 6483.79$ km	$z_2 = 4911.73$ km

$\mu = 398600.8\ \text{km}^3/\text{sec}^2$

First: The value of the semi-major axis to accuracy $\approx 10^{-2}$ meter was found to be $a = 7209.977140084$ km with $n = 16$

Second: The other orbital elements are:

$e = 0.061080$; $i = 65^0.1132$; $\Omega = 114^0.8612$; $\omega = 277^0.1397$; $u = 110^0.6302$.

2. Minor Planet 1569 Evita (Sconzo, 1962)

$t_1 = 0$

$t = t_2 = 28.9118$ days

$x_1 = 2.376754$ AU	$x_2 = 2.507401$ AU
$y_1 = -1.102329$ AU	$y_2 = -0.826966$ AU
$z_1 = -0.973496$ AU	$z_2 = -0.896717$ AU

$\mu = 0.000295912\ \text{AU}^3/\text{day}^2$

First: The value of the semi-major axis to accuracy $\approx 10^{-11}$ AU was found to be

$a = 3.1568626454$ AU with $n = 17$

Second: The other orbital elements are:
$e = 0.117682$; $i = 24^0.2635$; $\Omega = 30^0.6399$; $\omega = 316^0.7239$; $u = 302^0.0492$.

3. Hyperbolic orbit

$t_1 = 0$
$t = t_2 = 1000$ s

$x_1 = -10316.00709$ km	$x_2 = -5081.722922$ km
$y_1 = -6389.956846$ km	$y_2 = -4306.977002$ km
$z_1 = -4005.124124$ km	$z_2 = -14234.301845$ km

$\mu = 398600.8\ \text{km}^3/\text{sec}^2$

First: The value of the semi-major axis to accuracy $\approx 10^{-11}$ AU was found to be $a = -5102.503477929$ km with $n = 17$

Second: The other orbital elements are:
$e = 3.49358$; $i = 85^0.330$; $\Omega = 30^0.23$; $\omega = 204^0.366$; $u = 198^0.3287$.

The data of the last illustration is obtined as follows:

- We already used the orbital elements of the illustration.
- From the applications of the standered trasformation formulae between the orbital elements and the position and velocity vectors with t=1000 sec. we obtained the coordinats (x_1, y_1, z_1) and the corresponding velocity vecor(4.452701327, 1.5666645370, -10.8730539400 km/sec).
- With these position and velocity vectors as initial values for the differential system of the pure Kepler motion we get from the application of Runge-Kutta method the coordinats (x_2, y_2, z_2) after 1000sec.

Finally, it should be mentioned that, all the results of the above numerical illustrations are in agreements with results from which their data were taken.

CONCLUSION

In concluding the present paper, an efficient orbit determination method was developed using Thorne's power series solution of Lambert problem. Its efficiency is due to factors such as:

- The used analytical power series is *invariant* under many operations because, addition, multiplication, exponentiation, arising to powers, differentiation, integration, etc. of a power series is also a power series. A fact which provides excellent flexibility in dealing with the analytical as well as computational developments of problems related to orbit determination.
- The method is universal in the sense that, it uses one algorithm for both elliptic and hyperbolic orbits. Importance of the universal formulations is due to the fact that: during space mission all types of the two body motion appear. For examples the escape from the departure planet and the capture by the target planet involve hyperbolic orbits, while the

intermediate stage of the mission commonly depicted as a heliocentric ellipse, may also be heliocentric hyperbola. In addition, in some systems, the type of an orbit is occasionally changed by perturbing forces during finite interval of time. Thus we have been obliged to use different functional representations for motion depending upon the energy state and a simulation code must then contain branching to handle a switch from one state to another .In cases where this switching is not smooth, branching can occur many times during a single integration time-step causing some numerical "chatter". Consequently, universal formulations are desperately needed so that, orbit determination will be free of the troubles, since a single functional representation suffices to describe all possible states.

- The method does not need the solution of Kepler's equation and its variants for hyperbolic orbits. A fact, which is very important, because these equations are transcendental and could be solved by iterative methods which in turn need: (a) initial guess, (b) an iterative scheme. In fact, these two points are not separated from each other, but there is a full agreement that, even accurate iterative schemes are extremely sensitive to initial guess. Moreover, in many cases the initial guess may lead to drastic situation between divergent and very slow convergent solutions (Sharaf et al. 2007).
- The method is of dynamical nature, in the sense that it includes iterative schemes, such that moving from one scheme to the subsequent one, only one additional instruction is needed. This recurrent nature facilitates the computations of any number of the series coefficients needed for accurate orbit determination.

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طريقة لتعين المدارات باستخدام متسلسلة حل ثورني (Thorne) لمسألة لمبرت

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الملخص

تم فى هذا البحث تشييد طريقة لتعين المدارات وذلك باستخدام متسلسلة حل ثورني (Thorne) لمسألة لمبرت. تميزت هذة الطريقة بأنها شاملة، بمعنى إنها تصلح للمدارات الأهيلجية والزائدية. كما أن الطريقة لا تستخدم الحلول التكرارية لمعادلة كبلر لكل من هذة المدارا ت الأمر البالغ الأهمية، حيث أن الحلول التكرارية تعتمد على تخمين ابتدائى والذى بدورة قد يؤدى إلى تباعد الحل أو على أحسن تقدير إلى حل بطئ التقارب جداً.