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On T-Equivariant Rational Equivalence of T-Equivariant Cycles

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ABSTRACT

Let $\mathbb{P}^m$ be the m-dimensional projective space over the field of complex numbers $\mathbb{C}$ acted on by the algebraic torus $T=(\mathbb{C}^*)^{m+1}$. Let X be an equivariantly embedded variety in $\mathbb{P}^m$ via the map φ, i.e., φ is an embedding and $t.\varphi(x) \in \varphi(X)$ whenever $t \in T$ and $x \in X$. We begin a study of T-equivariant rational equivalence of T-equivariant cycles on a B-variety X, namely a non-singular projective variety with an action of a torus $T=(\mathbb{C}^*)^{m+1}$ with many fixed points. First, we compute the T-equivariant Picard group of such a variety with the help of fixed point and their associated characters. This result is then applied to study more generally T-equivariant rational equivalence of T-equivariant cycles.

INTRODUCTION

The diagonal group $G_n = \{g \in GL_n\mathbb{C} : g_{ij}=0 \text{ for } i \neq j\}$ is a closed subgroup of the general linear group $GL_n\mathbb{C}$ of invertible matrices over $\mathbb{C}$, which is evidently isomorphic to $(GL_1\mathbb{C})^n$. An algebraic group isomorphic to G_n is called an n-dimensional torus. Throughout this manuscript note that T will denote an $(m+1)-$ dimensional torus. The projective space $\mathbb{P}^m$ over $\mathbb{C}$ admits a natural action of the algebraic torus $T=(\mathbb{C}^*)^{m+1}$. The multiplicative one-parameter subgroups in T are the elements of $X_*(T)=\mathrm{Hom}(\mathbb{C}^*,T)$. Since the character group $X(T)=\mathrm{Hom}(T,\mathbb{C}^*)$, we compose to obtain the dual pairing $X(T)\times X_*(T)\to\mathbb{Z}$ given by $<\chi,\mu>=(\chi\circ\mu)(t)=t^k$.

We say that X is an equivariantly embedded variety in $\mathbb{P}^m$ if there is an embedding φ such that $t.\varphi(x)\in\varphi(X)$ whenever $x\in X$ and $t\in T$. Throughout this paper we assume that X is an equivariantly embedded B-variety in $\mathbb{P}^m$ via a fixed embedding. Fix a system of homogeneous coordinates $x_0,\dots,x_m$. Let ρ_i be the character of T defined by $\rho_i(t_0,\dots,t_n)=t_i$. Given the characters $\chi_0,\dots,\chi_n$ where $\chi_j=\Pi_{i=0}^m\rho_i^{n_{ij}}$, $n_{ij}\in\mathbb{Z}$. Identify X with its image $\varphi(X)$. Then T acts on X via $t.x_j=\chi_j^{-1}(t)x_j$, and on points $(a_0,\dots,a_m)$, this action is given by

$$t.(a_0,\ldots,a_m) = (\chi_0(t)a_0,\ldots,\chi_m(t)a_m).$$

If $f: X \to \mathbb{P}^1$ is a rational function on X then T acts on $f(x)$ via

$$t.f(x) = f(t^{-1}.x) = f(\chi_0{}^{-1}(t)x_0,\ldots,\chi_m{}^{-1}(t)x_m).$$

This paper is devoted to characterize T-equivariant rational equivalence of T-equivariant cycles on a B-variety X. A cycle on an arbitrary algebraic variety (or scheme) X is a finite formal sum $\sum n_V[V]$ of irreducible subvarieties of X, with integer coefficients. A rational function f on any subvariety of X determines a cycle $[(f)]$ of the principal divisor (f). Cycles differing by a sum of such cycles are defined to be rationally equivalent. The group of rational equivalence classes on X is denoted A_*X (see (Fulton, 1984) Chap. 1). The group A_*X will play a roll analogous to homology groups in topology. When X is non-singular, $A^*X \simeq A_*X$; in the non-singular case, but not in general, A_*X will have a ring structure. The ring A_*X is often called the Chow ring of X. The actual relation of these groups to homology groups is discussed in Chapter 19 of (Fulton, 1984) .

In (Edidin and Graham, 1998), Edidin and Graham introduced Chow groups of a scheme X and they proved the localization theorem for torus actions in equivariant intersection theory. On the other hand, in (Edidin and Graham, 1998), they introduced definition and basic properties of equivariant Chow groups of a scheme and more generally algebraic spaces acted on by linear algebraic groups. These are algebraic analogues of equivariant cohomology groups which have all the functorial properties of ordinary Chow groups. In addition, they enjoy many of the properties of equivariant cohomology. The definition of that paper is modeled after Borel's definition of equivariant cohomology.

Let S be the tautological bundle on $\mathbb{CP}^\infty$ whose sheaf of sections is $\mathcal{O}_{\mathbb{CP}^\infty}(-1)$, and let $BT = (\mathbb{CP}^\infty)^n$. The principal T-bundle ET over BT is defined to be $ET = \pi_1^*S \oplus \ldots \oplus \pi_n^*S$ where $\pi_i : BT \to \mathbb{CP}^\infty$ is the i^{th} projection map. If now X is a topological space with a T-action, put $X_T = ET \times_T X$, which is it self a fiber bundle over BT with fiber X. The equivariant cohomology of X is defined to be $H_T^*(X) = H^*(X_T)$ where $H^*(X_T)$ is the ordinary cohomology of X_T . Let $\pi : X_T \to BT$ be the equivariant projection map induced by the trivial map $X \to point$. Let $\mathbb{C}$ be the constant sheaf on X_T . The key lemma of this work, namely Lemma 2.13 paves the way to prove the main result of this work. It indicates that the sheaf $\mathcal{R}^2\pi_*\mathbb{C}$ on BT is a direct sum of k copies of the constant sheaf $\mathbb{C}$ where k is the dimension of the vector space $H^2(X,\mathbb{C})$.

Consider the q-th direct image sheaf $\mathcal{R}^q\pi_*\mathbb{C}$ on BT associated to the presheaf $U \mapsto H^q(\pi^{-1}(U),\mathbb{C})$ where U is an open set in BT. The $Leray$ spectral sequence, is a spectral sequence $\{\mathcal{E}_r\}$ with

$$\left\{ \begin{array}{c} \mathcal{E}_\infty \Rightarrow H^*(X_T,\mathbb{C}), \\ \mathcal{E}_2^{p,q} = H^p(BT,\mathcal{R}^q\pi_*\mathbb{C}) \\ \end{array} \right\}.$$

This sequence degenerates i.e., $\mathcal{E}_\infty^{p,q} = \mathcal{E}_2^{p,q}$, $p,q \geq 0$ and we obtain the following filteration of $H^2(\mathbb{C})$

$$H^2(\mathbb{C}) = F^0 H^2(\mathbb{C}) \supset F^1 H^2(\mathbb{C}) \supset F^2 H^2(\mathbb{C}) \supset \ldots \supset F^n H^2(\mathbb{C}) \supset F^{n+1} H^2(\mathbb{C}) = 0$$

such that $F^p H^2(\mathbb{C})/F^{p+1} H^2(\mathbb{C}) \simeq \mathcal{E}_\infty^{p,2-p}$. For simplicity we let F^p denote for $F^p H^2(\mathbb{C})$. Then $F^0/F^1 = \mathcal{E}_\infty^{0,2} = H^0(BT,\mathcal{R}^2\pi_*\mathbb{C})$ and $F^1/F^2 = \mathcal{E}_\infty^{1,1} = H^1(BT,\mathcal{R}^1\pi_*\mathbb{C}) = 0$ because $\mathcal{R}^1\pi_*\mathbb{C} = 0$. But $F^1 \neq 0$ implies $F^1 = F^2$. Also $F^2/F^3 = \mathcal{E}_\infty^{2,0} = H^2(BT,\mathcal{R}^0\pi_*\mathbb{C}) = H^2(BT,\mathbb{C})$ and $F^3/F^4 = \mathcal{E}_\infty^{3,-1} = 0$ .Thus $F^3 = 0$ and by a similar argument $F^4 = F^5 = \ldots = 0$. Therefore, we obtain the filteration $F^0 \supset F^1 = F^2$. This gives rise to the short exact sequence

$$0 \to F^2 \hookrightarrow F^0 \to F^0/F^2 \to 0$$

Thus we get the exact sequence

$$0 \to H^2(BT,\mathbb{C}) \hookrightarrow^{\pi^*} H^2(X_T,\mathbb{C}) \to^{\psi} H^0(BT,\mathcal{R}^2\pi_*\mathbb{C}) \to 0$$

This was crucial in the proof of the key lemma.

Finally, in (Edidin and Graham, 1998), Edidin and Graham have computed more generally T-equivariant Chow groups of any variety with T-action via a localization formula. However, the approach taken here is more straightforward. We characterize T-equivariant rational equivalence of T-equivariant cycles on a B-variety X using the torus action and weights. Again the methods of the present paper are simpler and the characterization obtained is more explicit.

T-EQUIVARIANT LINEAR EQUIVALENCE IN A B-VARIETY

The purpose of this section is to determine the T-equivariant Picard group of a B-variety X, namely the group $H_T^2(X)$. Equivariant cohomology enjoys many of the usual properties of ordinary cohomology such as the existence of flat equivariant pullbacks and proper equivariant pushforwards. These maps play an important role in describing T-equivariant linear equivalence. Note that $H_T^*(point) = H^*(BT)$. By pullback via $X \to point$, we see in genenral that $H^*(X_T)$ is an $H^*(BT)-$ module. Thus $H^*(BT)$ may be regarded as the coefficient ring for equivariant cohomology.

We let $M(T)$ be the character group of the torus T. For each $\chi \in M(T)$, we get a 1-dimensional vector space $\mathbb{C}_\chi$ with a T-action given by χ. If $L_\chi = (\mathbb{C}_\chi)_T$ is the corresponding line bundle over BT, then the assignment $\chi \mapsto -c_1(L_\chi)$ defines an isomorphism $\upsilon : M(T) \to H^2(BT)$, which is in turn induces a ring isomorphism $Sym(M(T)) \simeq H^*(BT)$. We call $\upsilon(\chi)$ the weight of χ. In particular, if ρ_i is the character of T defined by $\rho_i(t_0, t_1, \ldots, t_m) = t_i$ then we let λ_i denote the weight of ρ_i, i=0,1,...,m. Thus we get the isomorphism $H^*_T(point) = H^*(BT) \simeq \mathbb{C}[\lambda_0, \ldots, \lambda_m]$.

For simplicity, we let λ_i denote $\pi^*\lambda_i$ in $H^*(X_T)$ where $\pi : X_T \to BT$ is the equivariant projection map induced by the trivial map $X \to$ point (Cox David and Katzt, 1999) and $\pi^* : H^*(BT) \to H^*(X_T)$ is the pullback map.

We denote the line bundle L_{ρ_i} by $\mathcal{O}(-\lambda_i)$, so that $\lambda_i = c_1(\mathcal{O}(\lambda_i))$. On the other hand, consider the action of T on $\mathbb{P}^m$ given by $(t_0, \ldots, t_m).(x_0, \ldots, x_m) = (t_0^{-1}x_0, \ldots, t_m^{-1}x_m)$. The inverses has been chosen so that $(t_0, \ldots, t_m)$ acts on the homogeneous form $x_j \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ as multiplication by t_j.

Remark 2.1 The fiber bundle $\mathbb{P}^m_T$ is the projectivization of the vector bundle $F_T = \oplus_{i=0}^m \mathcal{O}(-\lambda_i)$ over BT which gives the tautological line bundle $\mathcal{O}_{\mathbb{P}^m_T}(1)$, and we have $p = c_1(\mathcal{O}_{\mathbb{P}^m_T}(1))$ belongs to $H^*_T(\mathbb{P}^m)$. We refer to p as the equivariant hyperplane class. Furthermore, the class $[(x_j = 0)_T] = p - \lambda_j$.

If X is a topological space with a T-action and $\mathcal{B} \to Y$ is a principal T-bundle, we recall that a fiber bundle $\mathcal{B} \times_T X$ is defined to be $\mathcal{B} \times_T X = (\mathcal{B} \times X)/((u, x) \sim (u \bullet t^{-1}, t \bullet x))$ for any $x \in X$, $t \in T$, and $u \in \mathcal{B}$.

Following Fulton ((Fulton, 1984) Example 1.9.1) we say that a scheme X has a cellular decomposition if there is a filtration $X = X_n \supset X_{n-1} \supset X_0 \supset X_{-1} = \phi$ by closed subschemes with each $X_i - X_{i-1}$ a disjoint union of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. The U_{ij}'s are called the cells of the decomposition.

Proposition 2.2((Ellingsrud and Strømme, 1987)) Let X be a scheme with a cellular decomposition. Then for $0 \leq i \leq$ dim X

(i) $H_{2i+1}(X) = 0$.

(ii) $H_{2i}(X)$ is a $\mathbb{Z}-$ module freely generated by the classes of the closure of the $i-$ dimensional cells.
(iii) The cycle map $cl: A_*(X) \to H_*(X)$ is an isomorphism.

The following lemma is useful. The proof is straightforward.

Lemma 2.3 Let X be a B-variety, and let $\mathbb{C}$ be the constant sheaf on X_T. Then the sheaf $\pi_*\mathbb{C}$ is isomorphic to the constant sheaf $\mathbb{C}$ where π_* is the pushforward induced by the equivariant projection map π.

Remark 2.4 We say that a subset $Y \subseteq X$ is T-equivariant if $t.y \in Y$ whenever $y \in Y$ and $t \in T$.

Definition 2.5 Let $V \subseteq ET$ be an open subset of ET. Let $\gamma : ET \to BT$ be a principal T-bundle. Let L_ρ be a line bundle over BT that corresponds to a character ρ. Let $\bar{g}: V \to \gamma^* L_\rho$ be a local section of the line bundle $\gamma^* L_\rho \subset ET \times \mathbb{C}_\rho$. Then $\bar{g}(y) \in (\gamma^* L_\rho)_y = \{y\} \times \mathbb{C}_\rho$ and $\bar{g}(y) = (y, g(y))$ where $g: V \to \mathbb{C}$ is a continuous function. We define the ring $\gamma^* L_\rho(V)$ to be the set of all continuous functions $g: V \to \mathbb{C}$. The sheaf $\gamma^* L_\rho$ is a sheaf of rings. We use $\gamma^* L_\rho$ to denote the line bundle associated to the sheaf of rings.

Definition 2.6 Let $V \subseteq X$ be a T-equivariant open subset of a B-variety X such that $V_T = ET \times_T V$, and let $[y, x]$ denote a class in V_T. We define a local section $\sigma : V_T \to \pi^* L_\rho$ by $\sigma([y, x]) = ([y, x], u_\sigma(x) g_\sigma(y))$ where $u_\sigma : V \to \mathbb{C}$ and $g_\sigma : ET \to \mathbb{C}$ are continuous functions. We define the ring $\pi^* L_\rho(V_T) := \{u(x)g(y) : u(x)$ is a continuous function from V to $\mathbb{C}$, and $g(y)$ is a continuous function from ET to $\mathbb{C}\}$. We use $\pi^* L_\rho$ to denote the line bundle associated to the sheaf of rings.

Remark 2.7 In algebraic geometry, divisors are a generalization of codimension one subvarieties of algebraic varieties; two different generalizations are in common use, Cartier divisors and Weil divisors. The concepts agree on non-singular varieties over algebraically closed fields.

To every Cartier divisor $D \subset X$ (X arbitrary variety or scheme) there is an associated line bundle (strictly, invertible sheaf) commonly denoted by $O_X(D)$, and the sum of divisors corresponds to the tensor product of line bundles. Isomorphism of bundles corresponds precisely to linear equivalence of Cartier divisors, and so the divisor classes give rise to elements in the Picard group. In otherwords, this defines a group morphism from the group of Cartier divisors modulo linear equivalence to the Picard group. This morphism is injective but is not always surjective. We are interested in the case where X is a B-variety.

Definition 2.8 Let $V \subseteq X$ be a T-equivariant open subset of a B-variety X such that $V_T = ET \times_T V$. Let $[y, x]$ denote a class in V_T. We define the ring

$O_{X_T}(D_T)(V_T) := \{h(x)g(y) : h(x)$ is a meromorphic function on V, and $g(y)$ is a continuous function on ET such that $(h) + D \geq 0$ and $h(tx)g(yt') = h(x)g(y)\}$. The sheaf $O_{X_T}(D_T)$ is a sheaf of rings. For simplicity, we use $O(D_T)$ to denote $O_{X_T}(D_T)$.

Definition 2.9 Let $V \subseteq X$ be a T-equivariant open subset of a B-variety X. Let $f: V \to \mathbb{P}^1$ be a rational function such that $t.f = \rho(t)f$. Let $f(0)$, $f(\infty)$ be the divisor of zeros and the divisor of poles of f respectively. Then the equivariant divisor $(f)_T = (f(0))_T - (f(\infty))_T$. We define $O((f)_T)(V_T)$ to be the ring $O((f)_T)(V_T) := \{\alpha : (V - f(\infty))_T \to \mathbb{C}$: $\alpha([y,x]) = g(y)f(x)$ where $g: ET \to \mathbb{C}$ is a continuous function associated to a section $\bar{g}$ of the line bundle $\gamma^* L_\rho$. We use $O((f)_T)$ to denote the line bundle associated to the sheaf of rings.

Lemma 2.10 Let $f: X \to \mathbb{P}^1$ be a rational function on a B-variety X. Then there exists a character ρ such that $t.f = \rho(t)f$.

Proof. Let $t \in T$ then $t.f$ and f have the same zeros and poles since $D - D'$ is T-equivariant. So $\frac{t.f}{f}$ has no zeros or poles in X where X is a compact set. Therefore $\frac{t.f}{f} = c_t$ where c_t is a constant. To check that c_t is a character, write $f(t^{-1}x) = (f \circ t^{-1})(x)$ where t^{-1} defines the bijection on X given by $x \mapsto t^{-1}x$. Now $c_{t.t'}f(x) = (t.t').f(x) = (f \circ (t^{-1}t'^{-1}))(x)$. It follows $c_{t.t'}f(x) = (f \circ t^{-1})(t'^{-1}x) = (c_t f)(t'^{-1}x) = c_t f(t'^{-1}x) = c_t c_{t'} f(x)$.

Proposition 2.11 Let $f: X \to \mathbb{P}^1$ be a rational function on a B-variety X. Then the sheaf $O((f)_T)$ is isomorphic to the sheaf $\pi^* L_\rho$ where $t.f(x) = \rho(t)f(x)$.

Proposition 2.12 Let $Y \subset X$ be a T-equivariant subvariety of X. Let $D_i \subset Y$ be a T-equivariant subvariety of Y such that $D_1 - D_2 = (f)$ where $f: Y \to \mathbb{P}^1$ is a rational function on Y. Then the restriction of $O(D_{1\,T})$ to Y_T is isomorphic to the restriction of $O(D_{2\,T})\pi^* L_{\rho'}$ to Y_T.

Proof. By Lemma 2.10 there exists a character ρ such that $t.f = \rho(t)f$. Let V be a T-equivariant open subset of Y. Then $V_T = ET \times_T V$. Let $[a,x]$ denote a class in V_T. Define the morphism

$$\varphi(V_T) : O(D_{2\,T})(V_T) \mid_{Y_T} \pi^* L_{\rho'}(V_T) \mid_{Y_T} \to O(D_{1\,T})(V_T) \mid_{Y_T}$$

by

$$\varphi(V_T)(h(x)g(a)k(x)l(a)) = k(x)h(x)f(x)g(a)l^2(a).$$

Now $(khf) + D_1 = (k) + (h) + D_2 \geq 0$, since $(k) \geq 0$ and $(h) + D_2 \geq 0$.

On the other hand, $h(tx)g(at^{t}) = h(x)g(a)$, $k(tx)l^{2}(at^{t}) = \rho'(t)k(x)l^{2}(a)$ and $f(tx) = \rho(t)f(x)$. Thus,

$$k(tx)h(tx)f(tx)g(at^{t})l^{2}(at^{t}) = k(x)h(x)f(x)g(a)l^{2}(a).$$

Therefore, $k(x)h(x)f(x)g(a)l(a) \in O(D_{1\,T})(V_T)$. The other details is a routine check.

Let X be a projective variety. So X is a noetherian integral separated scheme which is regular in codimension one. We recall that two divisors D and D' are said to be linearly equivalent, written $D \sim D'$, if $D - D'$ is a principal divisor((Hartshorne, 1977)). If X is equivariantly embedded in $\mathbb{P}^m$ via the embedding φ , $\varphi(D)$ and $\varphi(D')$ are T-equivariant subsets of $\varphi(X)$ then the linear equivalence is called T-equivariant linear equivalence.

On the other hand, two cycles $[D]$ and $[D']$ are said to be linearly equivalent, written $[D] \sim [D']$, if $[D - D']$ is a cycle of a principal divisor. If X is equivariantly embedded in $\mathbb{P}^m$ via the embedding φ , $\varphi(D)$ and $\varphi(D')$ are T-equivariant subsets of $\varphi(X)$ then the linear equivalence is called T-equivariant linear equivalence. Furthermore, the cycles are called T-equivariant cycles.

Recall the map $i_X^* : H_T^*(X) \to H^*(X)$ which is defined by $i_X^*([Z_T]) = [Z]$ where Z is a T-equivariant subvariety of a B-variety X. Then $i_X^*(\lambda_j) = 0$. On the other hand, X has a cellular decomposition and the k-dimensional vector space $H^2(X, \mathbb{C})$ over $\mathbb{C}$ is generated by the set $\{[D_1], \ldots, [D_k]\}$ of classes of the closure of the one-dimensional cells of X (see (Ellingsrud and Strømme, 1987) ,(Fulton, 1984)). Furthermore, $D_i \subset X$ is a T-equivariant subvariety of X.

Let $\hat{\mathcal{F}} = \mathcal{R}^2\pi_*\mathbb{C}$ be the sheaf associated to the presheaf $\mathcal{F}$ on BT where $\mathcal{F}(U) = H^2(\pi'(U), \mathbb{C})$, U open in BT . The following key lemma paves the way to prove the main result of this section.

Lemma 2.13. Let X be a B-variety, and let $\mathbb{C}$ be the constant sheaf on X_T . Then $\mathcal{R}^2\pi_*\mathbb{C}$ is isomorphic to a direct sum $\mathbb{C}^k$ of k copies of $\mathbb{C}$ where k is the dimension of $H^2(X, \mathbb{C})$.

Proof. Fix a T -equivariant divisor $D \subset X$. Then $D_T \subset X_T$ is an equivariant divisor . Let $\mathcal{L}_{D_T}$ denote the line bundle $O(D_T)$ on X_T associated to the divisor D_T . Then $c_1(\mathcal{L}_{D_T} \mid_{\pi'(U)}) \in H^2(\pi'(U), \mathbb{C}) = \mathcal{F}(U)$ where U is an open subset of BT . Let $s_U = c_1(\mathcal{L}_{D_T} \mid_{\pi'(U)})$. Let V be an open subset of U , consider the restriction map $\varepsilon_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ which is defined by $\varepsilon_V^U(s_U) = \theta^* s_U$ where $\theta : \pi'(V) \hookrightarrow \pi'(U)$ is the inclusion map. Then $\varepsilon_V^U(s_U) = s_V$. Therefore we get a global section $\hat{D} \in \mathcal{F}(BT) = H^2(X_T, \mathbb{C})$. Consider the exact sequence

$$0 \to H^2(BT,\mathbb{C}) \hookrightarrow^{\pi^*} H^2(X_T,\mathbb{C}) \to^{\psi} H^0(BT,\mathcal{R}^2\pi_*\mathbb{C}) \to 0$$

Define $\phi : H^2(X,\mathbb{C}) \to H^0(BT,\mathcal{R}^2\pi_*\mathbb{C})$ by $\phi([D]) = \tilde{L}$ where $\tilde{D} = \psi(\hat{D})$. We show that ϕ is an isomorphism.

First, we check that ϕ is well-defined. Suppose $[D]$ and $[D']$ are T-equivariant linear equivalent cycles. Then L is T-equivariant linear equivalent to D'. So there exists a rational function $f: X \to \mathbb{P}^1$ such that $D - D' = (f)$. By Lemma 2.10, there exists a character ρ such that $t.f = \rho(t)i$. Now we show that $\hat{D} - \hat{D}' = -\sum_{i=0}^m a_i\pi^*\lambda_i$ where $a_i \in \mathbb{C}$:

Let $\{U_i\}$ be an open cover for BT. For simplicity, we let i^* denote the pullback map $i^*_{\pi'(U_i)} : H^*(X_T,\mathbb{C}) \to H^*(\pi'(U_i),\mathbb{C})$ of the inclusion map $i_{\pi'(U_i)} : \pi'(U_i) \to X_T$. Then $\varepsilon^{BT}_{U_i}(\hat{D}' - \hat{D}) = i^*(c_1(\mathcal{L}_{D'_T}) - c_1(\mathcal{L}_{D_T}))$. Thus, by Proposition 2.12 we have $\varepsilon^{BT}_{U_i}(\hat{D}' - \hat{D}) = i^* c_1(O(D'_T - D_T)) = i^* c_1(O((f)_T))$. But $O((f)_T) \simeq \pi^* L_\rho$ and $\pi^* L_\rho = O(-\sum_{i=0}^m a_i\lambda_i)$ where $\sum_{i=0}^m a_i\lambda_i$ is the weight of the character ρ. It follows $\varepsilon^{BT}_{U_i}(\hat{D}' - \hat{D}) = -\sum_{i=0}^m a_i\lambda_i$ since i^* is a $\mathbb{C}[\lambda_0,\ldots,\lambda_m]-$ module homomorphism. Therefore, $\varepsilon^{BT}_{U_i}(\hat{D}' - \hat{D} + \sum_{i=0}^m a_i\lambda_i) = 0$ for all i implies $\hat{D}' - \hat{D} + \sum_{i=0}^m a_i\lambda_i = 0$ (sheaf axiom). It follows that $\tilde{D}' - \tilde{D} = \psi(-\sum_{i=0}^m a_i\pi^*\lambda_i) = -\sum_{i=0}^m a_i(\psi \circ \pi^*)(\lambda_i) = 0$ since the sequence above is exact. Therefore ϕ is well-defined.

Second, we show that ϕ is injective. Suppose that $\phi([D]) = 0$. But the sequence is exact, so $\hat{D} = \sum_{i=0}^m a_i\lambda_i$ where $a_i \in \mathbb{C}$ and $\varepsilon^{BT}_{U_i}(\hat{D} - \sum_{i=0}^m a_i\lambda_i) = 0$. So $c_1(\mathcal{L}_{D_T} \mid_{\pi'(U)}) - \sum_{i=0}^m a_i\lambda_i = 0$. If $j : \pi'(U) \hookrightarrow \pi'(BT)$ is the inclusion map then $j^*(c_1(\mathcal{L}_{D_T}) - \sum_{i=0}^m a_i\lambda_i) = 0$. So $j^*(c_1(O(D_T - \sum_{i=0}^m a_i\lambda_i))) = 0$. Thus $[D_T] - \sum_{i=0}^m a_i\lambda_i$ is linearly equivalent to zero. Applying the map i^*_X we get $[D] = 0$. So ϕ is injective. Third, ϕ is surjective because $\phi \circ i^*_X = \psi$ and ψ is surjective.

Finally, ϕ is a ring homomorphism follows immediately from the definition of ϕ.

Theorem 2.14. Let X be a B-variety with a fixpoint set $X^T = \{p_j\}_{j=1}^n$, $n \le m+1$. Let $D_i \subset X$ be a $T-equivariant$ subvariety of codimension one, $i = 1,2$. Then

- $D_1 \sim D_2 \Leftrightarrow [D_{1T} - D_{2T}] \in Span\{\lambda_i : i = 0,1,\ldots,m\}$, where $Span\{\lambda_i : i = 0,1,\ldots,m\} = \{\sum_{i=0}^m c_i\lambda_i : c_i \in \mathbb{C}\}$
- Let $i_j^* : H^*_T(X) \to H^*_T(p_j)$ be the pullback induced by the equivariant inclusion

$i_{jT} : p_{jT} \hookrightarrow X_T$. If $[D_T]$ is any cycle class of an equivariant divisor such that $i_j^*([D_T]) = \sum_{i=0}^m a_i\lambda_i$, for all j, then $[D_T] = \sum_{i=0}^m a_i\lambda_i$.

Proof.

- We prove one direction. The other direction is straight forward. Suppose that $D_1 \sim D_2$ then $[(D_1 - D_2)_T] \in \ker i_X^*$. We show that $\ker i_X^* =$ Span $\{\lambda_i : i = 0, 1, \ldots, m\}$. Recall from Lemma 2.13 the isomorphism of k-dimensional spaces $\phi : H^2(X, \mathbb{C}) \to H^0(BT, \mathcal{R}^2\pi_*\mathbb{C})$. Let $\varsigma = \phi^{-1}$ then $\varsigma \circ \psi = i_X^*$ and ς is an isomorphism of k-dimensional spaces. Furthermore, $H^2(BT, \mathbb{C}) =$ Span $\{\lambda_i : i = 0, 1, \ldots, m\}$. Also note that $i_X^* \circ \pi^* = 0_{map}$, where 0_{map} is the zero map since $i_X^*(\lambda_j) = 0$.

  Now we show that ς is injective $\Leftrightarrow \ker i_X^* = H^2(BT, \mathbb{C})$.
  ($\Leftarrow$): Suppose $\varsigma(s) = 0$ where $s \in H^0(BT, \mathcal{R}^2\pi_*\mathbb{C})$. Since ψ is surjective there exists $w \in H^2(X_T, \mathbb{C})$ such that $\psi(w) = s$. It follows $\varsigma(\psi(w)) = \varsigma(s) = 0$. But $\varsigma \circ \psi = i_X^*$ implies $i_X^*(w) = 0$ which implies $w = \sum_{i=0}^m b_i\lambda_i$, $b_i \in \mathbb{C}$. It follows $s = \psi(w) = \psi(\sum_{i=0}^m b_i\pi^*(\lambda_i)) = \sum_{i=0}^m b_i\psi(\pi^*(\lambda_i)) = \sum_{i=0}^m b_i.0 = 0$. Hence ς is injective.

  $(\Rightarrow)$: Let $\alpha \in \ker i_X^*$ then $0 = i_X^*(\alpha) = \varsigma(\psi(\alpha))$. But ς is injective implies $\psi(\alpha) = 0$ which implies $\alpha \in \ker\psi = \mathrm{im}\pi^*$. Thus $\alpha \in H^2(BT, \mathbb{C})$. Hence $\ker i_X^* \subset H^2(BT, \mathbb{C})$. On the other hand $\ker i_X^*$ contains $H^2(BT, \mathbb{C})$ since $i_X^* \circ \pi^* = 0_{map}$. Therefore $\ker i_X^* = Span\{\lambda_i : i = 0, \ldots, m\}$.

- Let $\mathbb{C}(\lambda) = \mathbb{C}(\lambda_0, \ldots, \lambda_m)$ be the field of fractions of $\mathbb{C}[\lambda_0, \ldots, \lambda_m]$. Consider the map $\phi : H_T^*(X) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda) \to \oplus_{j=1}^n H_T^*(p_j) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$ where $\phi(\alpha \otimes f(\lambda)) = (i_j^*(\alpha).f(\lambda))_{j=1}^n$. Let $[D_T] \in H_T^2(X) \subset H_T^2(X) \otimes \mathbb{C}(\lambda)$ such that for each j we have $i_j^*([D_T]) = f(\lambda)$ where $f(\lambda) = \sum_{i=0}^m a_i\lambda_i$. Then $\phi(f(\lambda) \otimes 1) = \phi([D_T] \otimes 1)$ and ϕ is injective implies $f(\lambda) \otimes 1 = [D_T] \otimes 1$ which implies $[D_T] = f(\lambda)$.

T-EQUIVARIANT RATIONAL EQUIVALENCE IN A B-VARIETY

In this section we investigate more generally T-equivariant rational equivalence in a B-variety X. The field of rational functions on X is denoted $R(X)$; the non-zero elements of this field form the multiplicative group $R(X)^*$.

Definition 3.1 Suppose that V and V' are subvarieties of X of dimension k. We say that V and V' are rationally equivalent if there are a finite number of $(k+1)-$ dimensional subvarieties

V_i of X, and $f_i \in R(V_i)^*$ such that $V - V' = \Sigma(f_i)$. If X is equivariantly embedded in $\mathbb{P}^m$ via the embedding φ, $\varphi(V)$ and $\varphi(V')$ are T-equivariant subsets of $\varphi(X)$, then the rational equivalence is called T-equivariant rational equivalence.

Definition 3.2 Suppose that V and V' are subvarieties of X of dimension k. We say that two k-cycles $[V]$ and $[V']$ are rationally equivalent if there are a finite number of $(k+1)-$dimensional subvarieties V_i of X, and $f_i \in R(V_i)^*$ such that $[V]-[V'] = \Sigma[(f_i)]$. If X is equivariantly embedded in $\mathbb{P}^m$ via the embedding φ, $\varphi(V)$ and $\varphi(V')$ are T-equivariant subsets of $\varphi(X)$, then the rational equivalence is called T-equivariant rational equivalence. The cycles are called T-equivariant cycles.

Theorem 3.3. Let X be an n-dimensional B-variety with a fixpoint set $X^T = \{q_j\}_{j=1}^d$. Let $Z \subset X$ be a $T-$ equivariant subvariety of dimension $k+1 < n$, and let $D_i \subset Z$ be a $T-$equivariant subvariety of dimension k, $i = 1,2$. Then

- If Z is irreducible then

$$D_1 \sim D_2 \text{ in } Z \Leftrightarrow [D_{1T} - D_{2T}] \in Span\{\lambda_i.[Z_T] : i = 0,1,\ldots,m\}$$

  where $Span\{\lambda_i.[Z_T] : i = 0,1,\ldots,m\} = \{\sum_{i=0}^m c_i\lambda_i.[Z_T] : c_i \in \mathbb{C}\}$.
- Let $i_j^* : H_T^*(X) \to H_T^*(q_j)$ be the pullback induced by the equivariant inclusion $i_{jT} : q_{jT} \hookrightarrow X_T$. If $D_1 \sim D_2$ in Z and Z is an irreducible subvariety of X, then there exists an $l(\lambda) = \sum_{i=0}^m a_i\lambda_i$, such that $i_j^*([D_{1T} - D_{2T}]) = l(\lambda) i_j^*([Z_T])$, $j = 1,\ldots,d$.
- Let $D_i \subset \cup_{k=1}^s Z_k$, $i = 1,2$, be a T-equivariant subvariety of dimension r, and let $Z_k \subset X$ be an irreducible T-equivariant subvariety of dimension $r+1 < n$. If D_1 and D_2 are T-equivariant rational equivalent, and $D_i = \sum_{k=1}^s D_{ik}$ such that $D_{1k} \sim D_{2k}$ in Z_k. Then there exists $l_k = \sum_{i=0}^m a_{ik}\lambda_i$ such that

$$i_j^*([D_{1T} - D_{2T}]) = \sum_{k=1}^s l_k . i_j^*([Z_{kT}])$$

- where $j = 1,\ldots,d$.

Proof.

- Suppose that $D_1 \sim D_2$ in Z where D_i, Z are T-equivariant subvarieties of X. Then $D_1 - D_2 = (f)$ where $f : Z \to \mathbb{P}^1$ is a rational function on Z. By Lemma 2.10 there exists a character ρ such that $t.f = \rho(t)f$. Now let $\sum_{i=0}^m a_i\lambda_i$ be the weight of the character ρ. Then $[D_{2T} - D_{1T}] = [(f)_T]$ follows from Proposition 2.11 and Proposition 2.12 above. But $[(f)_T] = c_1(\pi^* L_\rho|_{Z_T}) = c_1(\pi^* L_\rho).[Z_T] = (-\sum_{i=0}^m a_i\lambda_i).[Z_T]$ which lives in Span $\{\lambda_i.[Z_T] : i = 0, 1, \ldots, m\}$. The other parts of the theorem follows easily.

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حول التكافؤ النسبي لمتساوي المتغير-T للدورات متساوية المتغير-T

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الملخص :

فلتكن P^n تمثل الفضاء الاسقاطي ذو الأبعاد-m على حقل الأعداد العقدية C المؤثر على الطارة(تورس) الجبرية $T=(C^*)^{m+1}$. نفرض x هو تنوع مطوق متساوي التغييرفي P^m عن طريق الإقتران φ ، أي أن φ مطوقة و $t.\varphi(x)\in\varphi(X)$ عندما تكون $t\in T$, $x\in X$. نبدأ بدراسة التكافؤ النسبي لمتساوي المتغير-T للدورات متساوية المتغير-T حول التنوع $-B$ ،بمعنى التنوع الإسقاطي الغير منفرد بتأثير الطارة $T=(C^*)^{n+1}$ باللإضافة إلى العديد من النقط الثابتة. نحسب أولاً مجموعة بيكادر لمتساوي المتغير-T بالاستعانة بنقطة ثابته وحصائصها الملازمة. تطبق هذه النتيجة لدراسة أعمم للتكافؤ النسبي لمتساوي المتغير T للدورات متساوية المتغير-T.