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طريقة إعادة إنتاج قلب فراغ هيلبرت لحل مسألة توتسش

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الملخص:

فى هذا البحث تم تطبيق طريقة إعادة توليد قلب فراغ هيلبرت لحل مسألة تورسش. لقد تم استخدام أمثلة عددية لتوضيح دقة وتطبيق هذه الطريقة. النتيجة التحليلية للمعادلة تم استنتاجها من خلال متسلسة تقاربية ولها معاملات بسيطة فى طريقة حسابها. نتائج الدراسة تمت مقارنتها بالنتائج التي تم الحصول عليها باستخدام طرق مختلفة مثل: اضطراب الهموتوبى وتحليلات لابلاس والاضطراب وتحليلات ادومين والتغاير المتكرر وأخيرا الطريقة الغير معيارية لنموذج الفروق المحدودة وذلك باستخدام جداول وأشكال. النتائج العددية بينت ان الطريقة الجديدة هي فعالة.

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ORIGINAL ARTICLE

The reproducing kernel Hilbert space method for solving Troesch's problem

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Abstract In this paper, the reproducing kernel Hilbert space method (RKHSM) is applied for solving Troesch's problem. We used numerical examples to illustrate the accuracy and implementation of the method. The analytical result of the equation has been obtained in terms of a convergent series with easily computable components. The results are compared with the ones obtained by the homotopy perturbation method (HPM), the Laplace decomposition method (LDM), the perturbation method (PM), the Adomian decomposition method (ADM), the variational iteration method (VIM), the B-spline method and the nonstandard finite difference scheme (FDS) by using tables and figures. Numerical results show that the present method is effective.
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1. Introduction

We consider in this work the boundary-value problem, Troesch's problem

$$u'' = \lambda \sinh(\lambda u), \quad 0 \leqslant x \leqslant 1, \tag{1.1}$$

with boundary conditions

$$u(0) = 0, \quad u(1) = 1, \tag{1.2}$$

where $u = u(x)$ and λ is a positive constant. This problem was described and solved by Weibel (1958). It arises from a system of nonlinear ordinary differential equations which occur in an investigation of the confinement of a plasma column by radiation pressure. Troesch (1976) found its numerical solution by the shooting method. The closed form solution to this problem in terms of the Jacobian elliptic function has been given (Roberts and Shipman, 1976) as

$$u(x) = \frac{2}{\lambda}\sinh^{-1}\left\{\frac{u'(0)}{2} sc\left[\lambda x, 1 - \frac{1}{4}(u'(0))^2\right]\right\}, \tag{1.3}$$

where $u'(0)$, the derivative of u at 0, is given by the expression $u'(0) = 2\sqrt{1-m}$, with m being the solution of the transcendental equation

$$\frac{\sinh\left(\frac{\lambda}{2}\right)}{\sqrt{1-m}} = sc(\lambda, m), \tag{1.4}$$

where $sc(\lambda, m)$ is the Jacobi elliptic function and m is a modulus of the jacobi elliptic function ($0 < m < 1$). From (1.3), it was noticed that a pole of $u(x)$ occurs at a pole of $sc\left[\lambda x, 1 - \frac{1}{4}(u'(0))^2\right]$. It was also noticed that the pole occurs at

$$x \approx \frac{1}{2\lambda}\ln\left(\frac{16}{1-m}\right). \tag{1.5}$$

This problem has been studied extensively. A numerical algorithm based on the decomposition method is presented by Deeba

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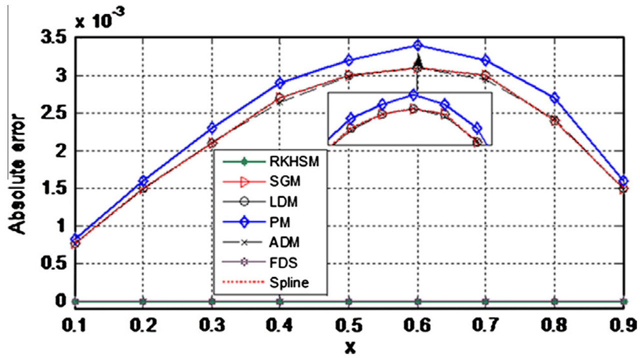

Figure 1 Compare the absolute errors previously obtained by various methods with our method for $\lambda = 0.5$.

et al. (2000), and they obtained the accurate numerical results using a few terms of the iterative scheme. Khuri (2003) used the Laplace transform and a modified decomposition technique for problem (1.1) and (1.2). Momani et al. (2006) implemented the variational iteration method for approximate and analytical solutions of boundary value problems. Feng et al. (2007) presented the modified homotopy perturbation method for Troesch's problem. Chang (2010) used the simple shooting method for this problem. Chang (2011) , in other work, proposed a variable transformation to solve Troesch's problem. The hyperbolic-type nonlinearity in the problem converted polynomial type nonlinearities and the transformed problem is solved by using the variational iteration method. Erdogan and Ozis (2011) presented a new kind of finite difference scheme for special second order nonlinear two-point boundary value problems. Geng and Cui (2011) solved nonlinear two-point boundary value problems by using a combination of the Adomian decomposition method (ADM) and the reproducing kernel method. Khuri and Sayfy (2011) used a finite element approach based on the cubic B-spline collocation method to solve problem (1.1) and (1.2). Hassan and El-Tawil (2011) used the homotopy analysis method (HAM) to solve two-point boundary value problems. Zarebnia and Sajjadian (2012) applied the sinc-Galerkin method (SGM) for solving Troesch's problem. Bougoffa and and Al-khadhi (2009) used New Explicit Solutions for Troesch's Boundary Value Problem.

In this paper, the RKHSM (Cui and Deng, 1986; Cui and Lin, 2009) will be used to investigate Troesch's problem. In recent years, a lot of attention has been devoted to the study of RKHSM to investigate various scientific models. The RKHSM which accurately computes the series solution is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that can be elegantly computed.

Recently, a lot of research work has been devoted to the application of RKHSM to a wide class of stochastic and deterministic problems involving fractional differential equation, nonlinear oscillator with discontinuity, singular nonlinear two-point periodic boundary value problems, integral equations and nonlinear partial differential equations and so on (Cui and Deng, 1986; Jiang and Cui (2009)).

The efficiency of the method was used by many authors to investigate several scientific applications. Geng and Cui (2007) applied the RKHSM to handle the second-order boundary value problems. Yao and Cui (2007) and Wang et al. (2008) investigated a class of singular boundary value problems by this method and the obtained results were good. Zhou et al. (2007) used the RKHSM effectively to solve second-order boundary value problems. In Lü and Cui (2008) the method was used to solve nonlinear infinite-delay-differential equations. Wang and Chao (2008), Li and Cui (2009) and Zhou and Cui (2009) independently employed the RKHSM to variable-coefficient partial differential equations. Geng and Cui (2010) and Du and Cui (2010) investigated the approximate solution of the forced Duffing equation with integral boundary conditions by combining the homotopy perturbation method and the RKHSM. Lv and Cui (2010) presented a new algorithm to solve linear fifth-order boundary value problems. In Jiang and Cui (2009) and Du and Cui (2010), authors developed a new existence proof of solutions for nonlinear boundary value problems. Cui and Du (2006) obtained the representation of the exact solution for the nonlinear Volterra–Fredholm integral equations by using the reproducing kernel space method. Wu and Li (2010) applied the iterative reproducing kernel method to obtain the analytical approximate solution of a nonlinear oscillator with discontinuties. Recently, the method was applied to the fractional partial differential equations and multi-point boundary value problems (Jiang and Lin, 2011; Mohammadi and Mokhtari, 2011). For more details about RKHSM and the modified forms and its effectiveness, see (Cui and Deng, 1986; Yao and Lin, 2011) and the references therein.

The paper is organized as follows. Section 2 is devoted to several reproducing kernel spaces and a linear operator is introduced. Solution representation in $W_2^3[a,b]$ has been presented in Section 3. Section 4 provides the main results, the exact and approximate solution of (1.1) and an iterative method are developed for the kind of problems in the reproducing kernel space. We have proved that the approximate solution converges to the exact solution uniformly. Some numerical experiments are illustrated in Section 5. We provide some conclusions in the last section.

2. Preliminaries

2.1. Reproducing kernel spaces

In this section, we define some useful reproducing kernel spaces.

Definition 2.1 (*Reproducing kernel*). Let E be a nonempty set. A function $K: E \times E \to \mathbb{C}$ is a reproducing kernel of the Hilbert space H if and only if

$$\begin{cases} \forall t \in E, \ K(.,t) \in H, \\ \forall t \in E, \forall \varphi \in H, \ \langle \varphi(.), K(.,t) \rangle = \varphi(t). \end{cases} \tag{2.1}$$

The last condition is called ''the reproducing property'' the value of the function φ at the point t is reproduced by the inner product of φ with $K(.,t)$.

Definition 2.2.

$$W_2^3[0,1] = \left\{ \begin{array}{l} u(x)|u(x), u'(x), u''(x), \text{ are absolutely continuous in } [0,1] \\ u^{(3)}(x) \in L^2[0,1], x \in [0,1], u(0) = 0, u(1) = 0. \end{array} \right\}$$

The inner product and the norm in $W_2^3[0,1]$ are defined respectively by

$$\langle u(x), g(x) \rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0) g^{(i)}(0) + \int_0^1 u^{(3)}(x) g^{(3)}(x) dx, u(x), g(x) \in W_2^3[0,1]$$

and

$$\|u\|_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}, \quad u \in W_2^3[0,1].$$

The space $W_2^3[0,1]$ is a reproducing kernel space, i.e., for each fixed $y \in [0,1]$ and any $u(x) \in W_2^3[0,1]$, there exists a function $R_y(x)$ such that
$u(y) = \langle u(x), R_y(x)\rangle_{W_2^3}$.

Definition 2.3.

$$W_2^1[0,1] = \left\{ \begin{array}{c} u(x)|u(x), \text{ is absolutely continuous in } [0,1] \\ u'(x) \in L^2[0,1],\ x \in [0,1], \end{array} \right\},$$

The inner product and the norm in $W_2^1[0,1]$ are defined respectively by

$$\langle u(x), g(x)\rangle_{W_2^1} = u(0)g(0) + \int_0^1 u'(x)g'(x)dx, u(x), g(x) \in W_2^1[0,1], \tag{2.2}$$

and

$$\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}, \quad u \in W_2^1[0,1]. \tag{2.3}$$

The space $W_2^1[0,1]$ is a reproducing kernel space and its reproducing kernel function $T_x(y)$ is given by

$$T_x(y) = \begin{cases} 1+x, & x \leqslant y, \\ 1+y, & x > y. \end{cases} \tag{2.4}$$

Theorem 2.1. *The space $W_2^3[0,1]$ is a complete reproducing kernel space and, its reproducing kernel function $R_y(x)$ can be denoted by*

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} c_i(y)x^{i-1}, & x \leqslant y, \\ \sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y, \end{cases}$$

where

$$c_1(y) = 0,$$
$$c_2(y) = \frac{5}{516}y^4 - \frac{1}{156}y^5 - \frac{5}{26}y^2 - \frac{5}{78}y^3 + \frac{3}{13}y,$$
$$c_3(y) = \frac{5}{624}y^4 - \frac{1}{624}y^5 + \frac{21}{104}y^2 - \frac{5}{312}y^3 - \frac{5}{26}y,$$
$$c_4(y) = \frac{5}{1872}y^4 - \frac{1}{1872}y^5 + \frac{7}{104}y^2 - \frac{5}{936}y^3 - \frac{5}{78}y,$$
$$c_5(y) = -\frac{5}{3744}y^4 + \frac{1}{3744}y^5 + \frac{5}{624}y^2 + \frac{5}{1872}y^3 - \frac{1}{104}y,$$
$$c_6(y) = \frac{1}{120} + \frac{1}{3744}y^4 - \frac{1}{18720}y^5 - \frac{1}{624}y^2 - \frac{1}{1872}y^3 - \frac{1}{156}y,$$
$$d_1(y) = \frac{1}{120}y^5,$$
$$d_2(y) = -\frac{1}{104}y^4 - \frac{1}{156}y^5 - \frac{5}{26}y^2 - \frac{5}{78}y^3 + \frac{3}{13}y,$$
$$d_3(y) = \frac{5}{624}y^4 - \frac{1}{624}y^5 + \frac{21}{104}y^2 + \frac{7}{104}y^3 - \frac{5}{26}y,$$
$$d_4(y) = \frac{5}{1872}y^4 - \frac{1}{1872}y^5 - \frac{5}{312}y^2 - \frac{5}{936}y^3 - \frac{5}{78}y,$$
$$d_5(y) = -\frac{5}{3744}y^4 + \frac{1}{3744}y^5 + \frac{5}{624}y^2 + \frac{5}{1872}y^3 + \frac{5}{156}y,$$
$$d_6(y) = -\frac{1}{156}y + \frac{1}{3744}y^4 - \frac{1}{18720}y^5 - \frac{1}{624}y^2 - \frac{1}{1872}y^3.$$

Proof. By Definition 2.2 we have

$$\langle u(x), R_y(x)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)R_y^{(i)}(0) + \int_0^1 u^{(3)}(x)R_y^{(3)}(x)dx. \tag{2.6}$$

Through several integrations by parts for (2.6) we have

$$\begin{aligned}\langle u(x), R_y(x)\rangle_{W_2^6} = {} & \sum_{i=0}^{2} u^{(i)}(0)\left[R_y^{(i)}(0) - (-1)^{(2-i)}R_y^{(5-i)}(0)\right] \\ & + \sum_{i=0}^{2}(-1)^{(2-i)}u^{(i)}(1)R_y^{(5-i)}(1) \\ & - \int_0^1 u(x)R_y^{(6)}(x)dx.\end{aligned} \tag{2.7}$$

Note that property of the reproducing kernel

$$\langle u(x),\ R_y(x)\rangle_{W_2^3} = u(y),$$

If

$$\begin{cases} R_y''(0) - R_y^{(3)}(0) = 0, \\ R_y'(0) + R_y^{(4)}(0) = 0, \\ R_y^{(3)}(1) = 0, \\ R_y^{(4)}(1) = 0, \end{cases} \tag{2.8}$$

then by (2.7) we have the following equation,

$$-R_y^{(6)}(x) = \delta(x-y),$$

When $x \neq y$,

$$R_y^{(6)}(x) = 0,$$

therefore

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} c_i(y)x^{i-1}, & x \leqslant y, \\ \sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y. \end{cases}$$

Since

$$-R_y^{(6)}(x) = \delta(x-y),$$

we have

$$\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0,1,2,3,4, \tag{2.9}$$

and

$$\partial^5 R_{y^+}(y) - \partial^5 R_{y^-}(y) = -1. \tag{2.10}$$

Since $R_y(x) \in W_2^3[0,1]$, it follows that

$$R_y(0) = 0, R_y(1) = 0, \tag{2.11}$$

From (2.8)–(2.11), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i = 1,2,\ldots,6)$ can be obtained. Thus $R_y(x)$ is given by

$$R_y(x)=\begin{cases}\frac{5}{516}xy^4-\frac{1}{156}xy^5-\frac{5}{26}xy^2-\frac{5}{78}xy^3+\frac{3}{13}xy+\frac{5}{624}x^2y^4-\frac{1}{624}x^2y^5+\frac{21}{104}x^2y^2\\ -\frac{5}{312}x^2y^3-\frac{5}{26}x^2y+\frac{5}{1872}x^3y^4-\frac{1}{1872}x^3y^5+\frac{7}{104}x^3y^2-\frac{5}{936}x^3y^3-\frac{5}{78}x^3y\\ -\frac{5}{3744}x^4y^4+\frac{1}{3744}x^4y^5+\frac{5}{624}x^4y^2+\frac{5}{1872}x^4y^3-\frac{1}{104}x^4y-\frac{1}{156}x^5y+\frac{1}{3744}x^5y^4\\ -\frac{1}{18720}x^5y^5-\frac{1}{624}x^5y^2-\frac{1}{1872}x^5y^3, \qquad x\leqslant y\\ \frac{5}{516}yx^4-\frac{1}{156}yx^5-\frac{5}{26}yx^2-\frac{5}{78}yx^3+\frac{3}{13}xy+\frac{5}{624}y^2x^4-\frac{1}{624}y^2x^5+\frac{21}{104}x^2y^2\\ -\frac{5}{312}y^2x^3-\frac{5}{26}y^2x+\frac{5}{1872}y^3x^4-\frac{1}{1872}y^3x^5+\frac{7}{104}y^3x^2-\frac{5}{936}x^3y^3-\frac{5}{78}y^3x\\ -\frac{5}{3744}x^4y^4+\frac{1}{3744}y^4x^5+\frac{5}{624}y^4x^2+\frac{5}{1872}y^4x^3-\frac{1}{104}y^4x-\frac{1}{156}y^5x+\frac{1}{3744}y^5x^4\\ -\frac{1}{18720}x^5y^5-\frac{1}{624}y^5x^2-\frac{1}{1872}y^5x^3, \qquad x>y\end{cases}$$

3. Solution representation in $W_2^3[0,1]$

In this section, the solution of (1.1) and (1.2) is given in the reproducing kernel space $W_2^3[0,1]$. On defining the linear operator $L: W_2^3[0,1]\to W_2^1[0,1]$ as

$$(Lu)(x)=u''(x). \tag{3.1}$$

Model problem (1.1) changes the following problem:

$$\begin{cases}Lu=f(x,u), & x\in[0,1]\\ u(a)=0, & u(b)=0,\end{cases} \tag{3.2}$$

Theorem 3.1. *The operator L defined by (3.1) is a bounded operator.*

Proof. We only need to prove $\|Lu\|^2_{W_2^1}\leqslant M\|u\|^2_{W_2^3}$, where $M>0$ is a given real constant. By (2.2) and (2.3), we have

$$\|Lu\|^2_{W_2^1}=\langle Lu,Lu\rangle_{W_2^1}=[(Lu)(0)]^2+\int_0^1[(Lu)'(x)]^2dx.$$

By reproducing property, we have

$$u(x)=\langle u(.),R_x(.)\rangle_{W_2^3},$$

and

$$(Lu)(x)=\langle u(.),(LR_x)(.)\rangle_{W_2^3},$$

so

$$|(Lu)(x)|\leqslant\|u\|_{W_2^3}\|LR_x\|^2_{W_2^3}=M_1\|u\|_{W_2^3}\quad(\text{where } M_1>0 \text{ is a positive constant}),$$

thus

$$(Lu)^2(0)\leqslant M_1^2\|u\|_{W_2^3}.$$

Since

$$(Lu)'(x)=\langle u(.),(LR_x)'(.)\rangle_{W_2^3},$$

then

$$|(Lu)'(x)|\leqslant\|u\|_{W_2^3}\|(LR_x)'\|^2_{W_2^3}=M_2\|u\|_{W_2^3}\quad(\text{where } M_2>0 \text{ is a positive constant}),$$

so, we have

$$[(Lu)'(t)]^2\leqslant M_2^2\|u\|^2_{W_2^3},$$

and

$$\int_0^1[(Lu)'(x)]^2dx\leqslant M_2^2\|u\|^2_{W_2^3},$$

that is

$$\|Lu\|^2_{W_2^1}\leqslant[(Lu)(0)]^2+\int_0^1[(Lu)'(x)]^2dx\leqslant\left(M_1^2+M_2^2\right)\|u\|_{W_2^3}=M\|u\|_{W_2^3},$$

where $M=M_1^2+M_2^2>0$ is a positive constant. □

4. The structure of the solution and the main results

In Eq. (3.1) it is clear that $L: W_2^3[0,1]\to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x)=T_{x_i}(x)$ and $\psi_i(x)=L^*\varphi_i(x)$, where L^* is a conjugate operator of L. The orthonormal system $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from the Gram-Schmidt orthogonaliztion process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\bar{\psi}_i(x)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x)\quad(\beta_{ii}>0,\quad i=1,2,\ldots) \tag{4.1}$$

Theorem 4.1. *For* Eq. (3.1), *if $\{x_i\}_{i=1}^{\infty}$ is dense on [0,1] then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$ and $\psi_i(x)=L_yR_x(y)|_{y=x_i}$.*

Proof. We have

$$\psi_i(x)=(L^*\varphi_i)(x)=\langle(L^*\varphi_i)(y),R_x(y)\rangle=\langle(\varphi_i)(y),LyR_x(y)\rangle=L_yR_x(y)|_{y=x_i}.$$

The subscript y by the operator L indicates that the operator L applies to the function of y. Clearly, $\psi_i(x)\in W_2^3[0,1]$. For each fixed $u(x)\in W_2^3[0,1]$, let $\langle u(x),\psi_i(x)\rangle=0$, $(i=1,2,\ldots)$, which means that,

$$\langle u(x),(L^*\varphi_i)(x)\rangle=\langle Lu(.),\varphi_i(.)\rangle=(Lu)(x_i)=0.$$

Note that, $\{x_i\}_{i=1}^{\infty}$ is dense on [0,1], hence, $(Lu)(x)=0$. From the existence of L^{-1} it follows that $u\equiv 0$. So the proof of Theorem 4.1 is complete. □

Theorem 4.2. *If $u(x)$ is the exact solution of* (3.2), *then*

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k,u_k)\overline{\Psi}_i(x), \tag{4.2}$$

where $\{x_i\}_{i=1}^{\infty}$ is a dense set in [0,1].

Proof. From the (4.1) and the uniqueness of solution of (3.2) we have

$$\begin{aligned}u(x)&=\sum_{i=1}^{\infty}\langle u(x),\bar{\psi}_i(x)\rangle_{W_2^3}\bar{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x),L^*T_{x_k}(x)\rangle_{W_2^3}\bar{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lu(x),T_{x_k}(x)\rangle_{W_2^1}\bar{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle f(x,u),T_{x_k}(x)\rangle_{W_2^1}\bar{\psi}_i(x)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k,u_k)\bar{\psi}_i(x).\end{aligned}$$

□

Table 1 The numerical results of example for boundary conditions at $\lambda = 0.1$.

x	Exact solution	Approximate solution	Absolute error	Relative error	Time
0.1	0.0998041752	0.0998041752	0.0	0.0	0.639
0.2	0.1996201980	0.1996201980	0.0	0.0	0.608
0.3	0.2994599206	0.2994599206	0.0	0.0	0.639
0.4	0.3993352030	0.3993352030	0.0	0.0	0.702
0.5	0.4992579186	0.4992579186	0.0	0.0	0.702
0.6	0.5992399580	0.5992399580	0.0	0.0	0.702
0.7	0.6992932326	0.6992932326	0.0	0.0	0.624
0.8	0.7994296804	0.7994296799	5×10^{-10}	6.25×10^{-10}	0.609
0.9	0.8996612690	0.8996612691	1×10^{-10}	1.11×10^{-10}	0.655
1.0	0.9999999996	1.0	4×10^{-10}	4.000×10^{-10}	0.593

Table 2 The numerical results of example for boundary conditions at $\lambda = 0.5$.

x	Exact solution	Approximate solution	Absolute error	Relative error	Time
0.1	0.09517690196	0.0951769018	7×10^{-11}	7.3547×10^{-10}	0.702
0.2	0.1906338690	0.1906338724	3.4×10^{-9}	1.7835×10^{-8}	0.826
0.3	0.2866534032	0.2866534036	4×10^{-10}	1.3954×10^{-9}	0.733
0.4	0.3835229288	0.3835229364	7.6×10^{-9}	1.98162×10^{-8}	0.858
0.5	0.4815373856	0.4815373906	5.0×10^{-9}	1.0383×10^{-8}	0.796
0.6	0.5810019748	0.5810019770	2.2×10^{-9}	3.7865×10^{-9}	0.827
0.7	0.6822351328	0.6822351353	2.5×10^{-9}	3.66442×10^{-9}	0.765
0.8	0.7855717868	0.7855717873	5×10^{-10}	6.36479×10^{-10}	0.843
0.9	0.8913669876	0.8913669937	6.1×10^{-9}	6.84342×10^{-9}	0.780
1.0	1.0	1.0	0.0	0.0	0.858

Table 3 The numerical results of example for boundary conditions at $\lambda = 1.0$.

x	Exact solution	Approximate solution	Absolute error	Relative error	Time
0.1	0.0817969965	0.0817965570	4.3956×10^{-7}	5.37379×10^{-6}	0.639
0.2	0.1645308708	0.1645307766	9.42×10^{-8}	5.72536×10^{-7}	0.639
0.3	0.2491673606	0.2491665307	8.299×10^{-7}	3.33069×10^{-6}	0.655
0.4	0.3367322088	0.336732458	2.492×10^{-7}	7.40053×10^{-7}	0.670
0.5	0.4283471608	0.4283474657	3.049×10^{-7}	7.11805×10^{-7}	0.733
0.6	0.5252740292	0.525275021	9.918×10^{-7}	1.88815×10^{-6}	0.655
0.7	0.6289711432	0.6289706684	4.748×10^{-7}	7.54883×10^{-7}	0.686
0.8	0.7411683772	0.7411684117	3.45×10^{-8}	4.6548×10^{-8}	0.702
0.9	0.8639700206	0.8639709620	9.414×10^{-7}	1.0896×10^{-6}	0.749
1.0	1.000000000	0.9999999997	3×10^{-10}	3.0000×10^{-10}	0.671

Table 4 The numerical results of example for boundary conditions at $\lambda = 1.5$.

x	Exact solution	Approximate solution	Absolute error	Relative error	Time
0.1	0.06364914304	0.0636491430	4×10^{-11}	6.2844×10^{-10}	0.702
0.2	0.1288746243	0.1288746242	1×10^{-10}	7.7594×10^{-10}	0.655
0.3	0.1973406481	0.1973406478	3×10^{-9}	1.5202×10^{-9}	0.717
0.4	0.2709010994	0.2709010994	0.0	0.0	0.671
0.5	0.3517328782	0.3517328831	4.9×10^{-9}	1.3931×10^{-9}	0.671
0.6	0.4425270074	0.4425270095	2.1×10^{-9}	4.7454×10^{-9}	0.639
0.7	0.5467812876	0.546781293	5.4×10^{-9}	9.8759×10^{-9}	0.686
0.8	0.6692735758	0.6692735734	2.4×10^{-9}	3.5859×10^{-9}	0.733
0.9	0.8168700768	0.816870073	3.8×10^{-9}	4.6519×10^{-9}	0.608
1.0	1.000000000	0.9999999996	4×10^{-10}	4.0000×10^{-10}	0.655

Table 5 The numerical results of example for boundary conditions at $\lambda = 2$.

x	Exact solution	Approximate solution	Absolute error	Relative error	Time
0.1	0.04584503564	0.0458450399	4.26×10^{-9}	9.2921×10^{-8}	0.608
0.2	0.09363547318	0.0936354708	2.38×10^{-9}	2.5417×10^{-8}	0.827
0.3	0.1454635787	0.1454635767	2.0×10^{-9}	1.3749×10^{-8}	0.639
0.4	0.2037490528	0.2037490524	4.1×10^{-10}	1.9631×10^{-9}	0.702
0.5	0.2715009814	0.2715009813	1.2×10^{-10}	3.6832×10^{-10}	0.733
0.6	0.3527451452	0.352745147	1.8×10^{-9}	5.1028×10^{-9}	0.577
0.7	0.4532840708	0.453284080	9.2×10^{-9}	2.0296×10^{-8}	0.671
0.8	0.5821576104	0.582157612	1.6×10^{-9}	2.7483×10^{-9}	0.639
0.9	0.7547145368	0.7547145379	1.1×10^{-9}	1.4575×10^{-9}	0.640
1.0	0.9999999996	1.000000006	6.4×10^{-9}	6.40000×10^{-9}	0.671

Table 6 The approximate solution of example for boundary conditions at $\lambda = 10$.

x	RKHSM	Time	[1], $\lambda=\mu=10$	MHPM [7]	VIM [6]	ADM [4]
			$s = 100x,\ n = 1$			
0.1	0.40	0.655	0.2452382346	17.61750000	0.1186109866	667081.18744
0.2	0.80	1.528	0.3046904881	33.69333333	0.4461962517	1333955.1189
0.3	−0.1	0.624	0.02841363154	46.78583333	3.8003366781	1999860.1189
0.4	1.0	0.609	−0.1977376785	55.65333333	79.891472730	2661970.7366
0.5	0.450751	2.043	−0.3864733430	59.35416667	1880.3539472	3310585.4201
0.6	−0.2	0.655	−0.05740959172	57.34666667	41642.365193	3914127.8659
0.7	0.78	0.624	0.1574986416	49.58916666	878764.64189	4374578.5342
0.8	0.941065	1.841	0.5230782099	36.64000000	18064027.967	4406724.4178
0.9	0.3	0.796	0.08763209033	19.75750000	366613074.02	3290268.6374
1.0	4.460660162	1.670	−0.1219698894	1.000000000	7396932871.8	1.0000000006

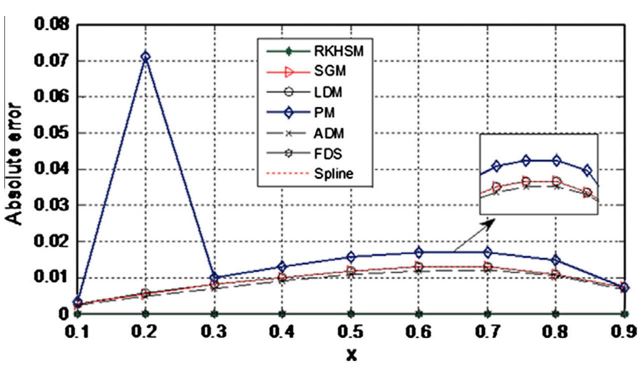

Figure 2 Comparing the absolute errors previously obtained by various methods with our method for $\lambda = 1$.

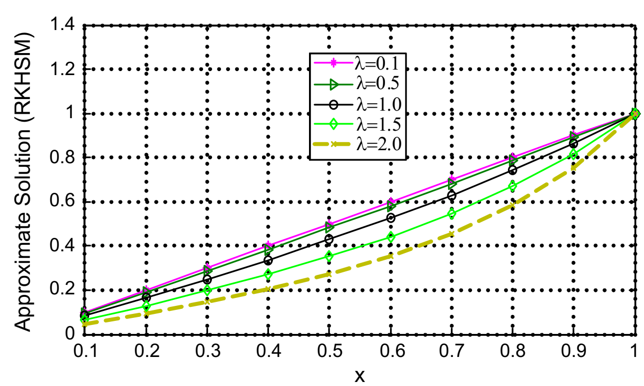

Figure 3 The reproducing kernel Hilbert solutions of Troesch's problem for $\lambda \leqslant 2$.

Now the approximate solution $u_n(x)$ can be obtained by truncating the n-term of the exact solution $u(x)$

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f(x_k,u_k)\bar{\psi}_i(x). \tag{4.3}$$

Lemma 4.1. *If* $u(x) \in W_2^3[0,1]$, *then there exists* $M_1 > 0$, *such that*

$$\|u\|_{C^2[0,1]} \leqslant M_1\|u\|_{W_2^3},$$

where $\|u\|_{C^2[0,1]} = \max_{x\in[0,1]}|u(x)| + \max_{x\in[0,1]}|u'(x)| + \max_{x\in[0,1]}|u''(x)|$.

Lemma 4.2. *If* $\|u_n - u\|_{W_2^3} \to 0, x_n \to x$, *(*$n\to\infty$*) and* $f(x,u)$ *is continuous for* $x \in [0,1]$, *then*

$$f(x_n, u_{n-1}(x_n)) \to f(x,u(x)) \text{ as } n \to \infty.$$

Proof. Since $\|u_n - u\|_{W_2^3} \to 0 (n\to\infty)$, by Lemma 4.1, we know $u_n(x)$ is convergent uniformly to $u(x)$, therefore, the proof is complete. □

Remark 4.1.

(i) If (3.2) is linear then the analytical solution of (3.2) can be obtained directly by (4.2).

(ii) If (3.2) is nonlinear then the solution of (3.2) can be obtained by the following iterative method.

We construct an iterative sequence $u_n(x)$, putting,

$$\begin{cases} \text{any fixed } u_0(x) \in W_2^3[0,1], \\ u_n(x) = \sum_{i=1}^{n} A_i \bar{\psi}_i(x), \end{cases} \tag{4.4}$$

where

$$\begin{cases} A_1 = \beta_{11} f(x_1, u_0(x_1)), \\ A_2 = \sum_{k=1}^{2} \beta_{2k} f(x_k, u_{k-1}(x_k)), \\ \ldots \\ A_n = \sum_{k=1}^{n} \beta_{nk} f(x_k, u_{k-1}(x_k)). \end{cases} \tag{4.5}$$

Next we will prove that $u_n(x)$ given by the iterative formula (4.4) is convergent to the exact solution (4.2).

Theorem 4.3. *Suppose the following conditions are satisfied: (i) $\|u_n\|_{W_2^3}$ is bounded; (ii) $\{x_i\}_{i=1}^{\infty}$ is dense in [0,1]; (iii) $f(x,u) \in W_2^1[0,1]$ for any $u(x) \in W_2^3[0,1]$. Then $u_n(x)$ in iterative formula (4.4) converges to the exact solution of (4.2) in $W_2^3[0,1]$ and*

$$u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i(x),$$

where A_i is given by (4.5).

Proof.

(i) First, we will prove the convergence of $u_n(x)$. By (4.4), we have

$$u_{n+1}(x) = u_n(x) + A_{n+1} \bar{\psi}_{n+1}(x).$$

From the orthogonality of $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\|u_{n+1}\|_{W_2^3}^2 = \|u_n\|_{W_2^3}^2 + (A_{n+1})^2 = \|u_{n-1}\|_{W_2^3}^2 + (A_n)^2 + (A_{n+1})^2$$
$$= \cdots = \sum_{i=1}^{n+1} (A_i)^2.$$

From boundedness of $\|u_n\|_{W_2^3}$, we have

$$\sum_{i=1}^{\infty} (A_i)^2 < \infty,$$

i.e.

$$\{A_i\} \in l^2 (i = 1, 2, \ldots).$$

Let $m > n$, in view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \ldots \perp (u_{n+1} - u_n)$, it follows that

$$\|u_m - u_n\|_{W_2^3}^2 = \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n\|_{W_2^3}^2$$
$$\leqslant \|u_m - u_{m-1}\|_{W_2^3}^2 + \ldots + \|u_{n+1} - u_n\|_{W_2^3}^2$$
$$= \sum_{i=n+1}^{m} (A_i)^2 \to 0 (m, n \to \infty).$$

Considering the completeness of $W_2^3[0,1]$, there exists $u(x) \in W_2^3[0,1]$, such that

$$u_n(x) \overset{\|\cdot\|_{W_2^3}}{\to} u(x), \text{ as } n \to \infty.$$

(ii) Second, we will prove $u(x)$ is the solution of (3.2).

By Lemma 4.1 and Theorem 4.3 (i), we know $u_n(x)$ convergence uniformly to $u(x)$. It follows that, on taking limits in (4.4),

$$u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i(x).$$

Since

$$(Lu)(x_j) = \sum_{i=1}^{\infty} A_i \langle L\bar{\psi}_i(x), \varphi_j(x) \rangle_{W_2^1} = \sum_{i=1}^{\infty} A_i \langle \bar{\psi}_i(x), L^* \varphi_j(x) \rangle_{W_2^3}$$
$$= \sum_{i=1}^{\infty} A_i \langle \bar{\psi}_i(x), \psi_j(x) \rangle_{W_2^3},$$

it follows that

$$\sum_{j=1}^{n} \beta_{nj} (Lu)(x_j) = \sum_{i=1}^{\infty} A_i \left\langle \bar{\psi}_i(x), \sum_{j=1}^{n} \beta_{nj} \psi_j(x) \right\rangle_{W_2^3}$$
$$= \sum_{i=1}^{\infty} A_i \langle \bar{\psi}_i(x), \bar{\psi}_n(x) \rangle_{W_2^3} = A_n.$$

If $n = 1$, then

$$(Lu)(x_1) = f(x_1, u_0(x_1)). \tag{4.6}$$

If $n = 2$, then

$$\beta_{21}(Lu)(x_1) + \beta_{22}(Lu)(x_2) = \beta_{21} f(x_1, u_0(x_1)) + \beta_{22} f(x_2, u_1(x_2)). \tag{4.7}$$

From (4.6) and (4.7), it is clear that

$$(Lu)(x_2) = f(x_2, u_1(x_2)).$$

Futhermore, it is easy to see by induction that

$$(Lu)(x_j) = f(x_j, u_{j-1}(x_j)). \tag{4.8}$$

Notice that $\{x_i\}_{i=1}^{\infty}$ is dense on interval [0,1], and $y \in [0,1]$, there exists a subsequence $\{x_{n_j}\}$, such that $x_{n_j} \to y$, as $j \to \infty$. Hence, let $j \to \infty$ in (4.8), by the convergence of $u_n(x)$ and Lemma 4.2 , we have

$$(Lu)(y) = f(y, u(y)),$$

that is, $u(x)$ is the solution of (3.2) and

$$u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i,$$

where A_i is given by (4.5). □

Corollary 4.1. *Assume that the conditions of* Theorem 4.3 *hold, then $u_n(x)$ in (4.4) satisfies $\|u_n - u\|_{C^2[0,1]} \to 0$, $n \to \infty$, where $u(x)$ is the solution of (3.2).*

Theorem 4.4. *Assume $u(x)$ is the solution of* Eq. (3.2) *and $r_n(x)$ is the error between the approximate solution $u_n(x)$ and the exact solution $u(x)$. Then the error sequence $r_n(x)$ is a monotone decreasing in the sense of $\|.\|_{W_2^3}$ and $\|r_n(x)\|_{W_2^3} \to 0$.*

Proof. From (4.2) and (4.3), it follows that,

$$\|r_n\|_{W_2^3} = \|\sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k,u_k)\bar{\psi}_i(x)\|_{W_2^3}$$
$$= \sum_{i=n+1}^{\infty}\left(\sum_{k=1}^{i}\beta_{ik}f(x_k,u_k)\right)^2 \quad (4.9)$$

(4.9) shows that the error r_n is decreasing in the sense of $\|\cdot\|_{W_2^3}$ □

5. Numerical results

In this section, numerical example is provided to show the accuracy of the present method for different values of λ. All computations are performed by Maple 13. Results obtained by the method are compared with the exact solution, and each λ values of the Adomian decomposition method (ADM) (Deeba et al., 2000), the Laplace decomposition method (LDM) (Khuri, 2003), the Variational iteration method (VIM) (Momani et al., 2006), the Modified homotopy perturbation method (MHPM) (Feng et al., 2007), the Nonstandard finite difference scheme (FDS) (Erdogan and Ozis, 2011), the B-spline collocation method (Khuri and Sayfy, 2011), and the sinc-Galerkin method (SGM) (Zarebnia and Sajjadian, 2012) are found to be in good agreement with each other. The RKHSM does not require discretization of the variables, i.e., time and space, it is not affected by the computation round off errors and one is not faced with the necessity of large computer memory and time. The numerical results we obtained justify the advantage of this methodology.

Remark 5.1. Bougoffa and Al-khadhi solved (1.1), (1.2) via $u = \mu\ (x)y$ and lie point symmetry. They obtained the following explicit solution of this problem.

$$y = \frac{1}{\mu(x)}\ln\left(\tan^2\left(\pm\frac{\sqrt{a}}{2}\int\mu^2(x)dx + nx + \frac{\pi}{4} \pm \frac{\sqrt{a}}{2}s(0)\right)\right), n = 0, 1, \ldots,$$

where

$$s(x) = \int\mu^2(x)dx, \mu_1 = \mu(1), \lambda(x) = a\mu^3(x), \lambda = \mu = 10, n = 1,$$
$$a = \frac{1}{100}, \tan^{-1}\left(\exp\frac{\mu_1}{2}\right) = \pm\frac{\sqrt{a}}{2}(s(0) + s(1)) + n\pi + \frac{n}{4},$$

using this explicit solution one can obtain different numerical results by arbitrary constants. In our study we found numerical results. Under the following conditions we compared our method with their method and both methods give stable results in Table 6.

We consider (1.1) and (1.2) for numerical results. After homogenizing the boundary conditions of (1.1) and (1.2) we obtain (5.1)

$$\begin{cases} y''(x) = \lambda\sinh(\lambda(y(x) + x)), 0 < x \leqslant 1, \\ y(0) = y(1) = 0. \end{cases} \quad (5.1)$$

Thus, if the method described above is applied to the (5.1) we find the following tables and figures.

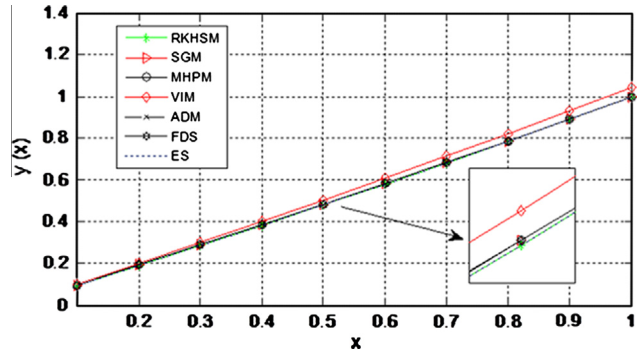

Figure 4 Comparing the numerical results previously obtained by various methods with our method and with the exact solution for $\lambda = 0.5$.

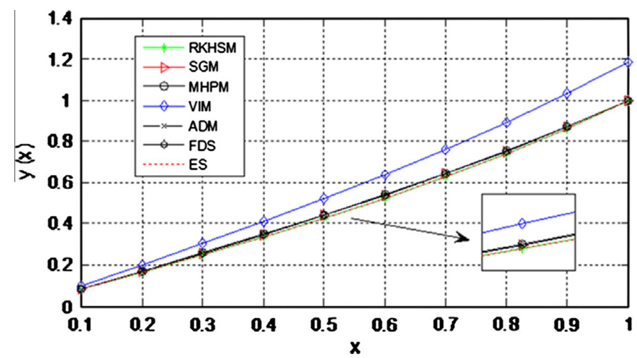

Figure 5 Comparing the numerical results previously obtained by various methods with our method and with the exact solution for $\lambda = 1$.

Tables 1–5 list the exact solution, approximate solution, absolute error, relative error and time corresponding to the various values of λ. As shown in the tables the method gave very good results for this problem. From semi-analytical and spline methods, the absolute error obtained is of around 10^{-3}, although by this method the absolute error obtained of around 10^{-8}. In Table 6 given results compared with other existing results. As shown in Table 6; MHPM, VIM and ADM methods gave unstable and divergent results. But our method and the method in Bougoffa and and Al-khadhi (2009) gave stable results. In Figs. 1 and 2, respectively, for $\lambda = 0.5$ and $\lambda = 1$ we compared the absolute error previously obtained by various methods with the absolute error that we obtained by this method. In Fig. 3 for $\lambda \leqslant 2$ the approximate solution obtained by RKHSM was given. In Figs. 4 and 5 respectively, for $\lambda = 0.5$ and $\lambda = 1$ we compared the numerical results previously obtained by various methods with the numerical results that we obtained by this method.

6. Conclusion

In this paper, we introduce an algorithm for solving Troesch's problem. For illustration purposes, we consider (1.1) and (1.2) for different values of λ which were selected to show the computational accuracy. It may be concluded that, the RKHSM is very powerful and efficient in finding an approximate solution for wide classes of problem. The approximate solution obtained by the present method is uniformly convergent.

Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems (Cui and Deng, 1986; Yao and Lin, 2011; Yang et al., 2008; Geng et al., 2010; Geng and Cui, 2007; Geng and Cui, 2012; Du and Geng, 2008; Wu and Li, 2011). However, if the problem becomes nonlinear, then the RKM does not require discretization or perturbation and it does not make closure approximation. Results of numerical examples show that the present method is an accurate and reliable analytical method for solving Troesch's problem.

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