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طريقة "هوموتوبي" التقاربية وتطبيقها لمعادلة (Kawahara) المعدلة

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الملخص:

يقدم البحث طريقة "هوموتوبي" المستحسنة التقاربية لإيجاد حل تقريبي لمعادلة (Kawahara) المعدلة. مقارنة هذه الطريقة مع طرق أخرى مثل: طريقة التكرار المتغير، طريقة الإضطراب الهوموتوبي، وكذلك الحلول المضبوطة تبين أن الطريقة المقدمة في هذا البحث ذات كفاءة جيدة.



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ORIGINAL ARTICLE

The optimal homotopy asymptotic method with application to modified Kawahara equation



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Abstract In this paper the optimal homotopy asymptotic method (OHAM) is introduced for obtaining the approximate solution of modified Kawahara equations. The OHAM results are compared with Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM) and Exact solutions. The comparison of OHAM with these methods reveals that OHAM is very effective, reliable and efficient.

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1. Introduction

The modified Kawahara equation has wide applications in physics such as plasma waves, capillary-gravity water waves, water waves with surface tension, shallow water waves (see Berloff and Howard, 1997; Hunter and Scheurle, 1998; Kawahara, 1972; Jin, 2009). The modified Kawahara equation has the form

$$\zeta_t + \zeta^2 \zeta_x + r \zeta_{xxx} + q \zeta_{xxxx} = 0, \quad (1)$$

where r , and q are nonzero real constants. The modified Kawahara equation has been solved by different analytic and numerical methods. These methods include the tanh-function method, extended tanh-function method, VIM, Sine–Cosine method, HPM, Jacobi elliptic function method and Adomian decomposition method (ADM) (see Sirendaoreji, 2004; Wazwaz, 2007; Yusufoglu and Bekir, 2006; Wazwaz, 2006;

Bibi and Mahyuddin, 2014; Noor et al., 2013; Zhang, 2005; Polat et al., 2006). The perturbation methods containing a small parameter and are difficult to be found were used for the solution of nonlinear boundary value problems (BVPs) (see O'Malley, 1974; Cole, 1968; Liu, 1997). The homotopy perturbation methods such as HPM, HAM and Artificial parameter method (see Liu, 1997) were introduced for finding the small parameter. Recently Marinca and Herisanu proposed OHAM for the solution of nonlinear BVPs (see Marinca et al., 2008, 2009; Herisanu et al., 2008; Herisanu and Marinca, 2010a,b). The authors have applied OHAM for obtaining the approximate solutions of nonlinear BVPs (see Islam and Shah, 2010; Idrees et al., 2010, 2012; Ullah et al., 2014; Nawaz et al., 2013). After introducing this method the perturbation methods become independent of the assumption of small parameter.

The first part of paper is introduction, and part 2 is devoted to the analysis of the proposed method. In part 3 solution of modified Kawahara equation is presented by OHAM and absolute errors with respect to Exact solution. The 3D and 2D images of the approximate solutions and exact solution are given. In all cases, the proposed method yields very encouraging results.

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Table 1 Comparison of absolute errors of OHAM solution to VIM, HPM and Exact solution at $x = -5$ for various values of t .

t	OHAM Sol	VIM Sol	HPM Sol	Exact Sol	Abs Errors
0	0.000947498	0.000947498	0.000947498	0.000947498	1.08420×10^{-19}
2	0.000947498	0.000947498	0.000947498	0.000946363	1.13562×10^{-6}
4	0.000947498	0.000947498	0.000947498	0.000944851	2.64695×10^{-6}
6	0.000947498	0.000947498	0.000947498	0.000942967	4.53160×10^{-6}
8	0.000947498	0.000947498	0.000947498	0.000940712	6.78656×10^{-6}
10	0.000947498	0.000947498	0.000947498	0.000938090	9.40828×10^{-6}

Table 2 Comparison of absolute errors of OHAM solution to VIM, HPM and Exact solution at $x = -2.5$ for different values of t .

t	OHAM Sol	VIM Sol	HPM Sol	Exact Sol	Abs Errors
0	0.000948387	0.000948387	0.000948387	0.000948387	1.08420×10^{-19}
2	0.000948387	0.000948387	0.000948387	0.000947723	6.63492×10^{-7}
4	0.000948387	0.000948387	0.000948387	0.000946682	1.70487×10^{-6}
6	0.000948387	0.000948387	0.000948387	0.000945264	3.12248×10^{-6}
8	0.000948387	0.000948387	0.000948387	0.000943473	4.91406×10^{-6}
10	0.000948387	0.000948387	0.000948387	0.000941310	7.07676×10^{-6}

Table 3 Comparison of absolute errors of OHAM solution to VIM, HPM and Exact solution at $x = 0$ for various values of t .

t	OHAM Sol	VIM Sol	HPM Sol	Exact Sol	Abs Errors
0	0.000948387	0.000948683	0.000948683	0.000948683	0
2	0.000948387	0.000948683	0.000948683	0.000948494	1.89711×10^{-7}
4	0.000948387	0.000948683	0.000948683	0.000947925	7.58542×10^{-7}
6	0.000948387	0.000948683	0.000948683	0.000946978	1.70558×10^{-6}
8	0.000948387	0.000948683	0.000948683	0.000945654	3.02932×10^{-6}
10	0.000948387	0.000948683	0.000948683	0.000943956	4.72765×10^{-6}

Table 4 Comparison of OHAM solution to VIM, HPM and Exact solution at $x = 2.5$ for various values of t .

t	OHAM Sol	VIM Sol	HPM Sol	Exact Sol	Abs Errors
0	0.000948387	0.000948387	0.000948387	0.000948387	1.08420×10^{-19}
2	0.000948387	0.000948387	0.000948387	0.000948671	2.84543×10^{-7}
4	0.000948387	0.000948387	0.000948387	0.000948577	1.89683×10^{-7}
6	0.000948387	0.000948387	0.000948387	0.000948102	2.84430×10^{-7}
8	0.000948387	0.000948387	0.000948387	0.000947250	1.13704×10^{-6}
10	0.000948387	0.000948387	0.000948387	0.000946020	2.36678×10^{-6}

Table 5 Comparison of OHAM solution VIM, HPM and Exact solution at $x = 5.0$ for various values of t .

t	OHAM Sol	VIM Sol	HPM Sol	Exact Sol	Abs Errors
0	0.000948387	0.000948387	0.000948387	0.000948387	1.08420×10^{-19}
2	0.000948387	0.000948387	0.000948387	0.000948671	2.84543×10^{-7}
4	0.000948387	0.000948387	0.000948387	0.000948577	1.89683×10^{-7}
6	0.000948387	0.000948387	0.000948387	0.000948102	2.84430×10^{-7}
8	0.000948387	0.000948387	0.000948387	0.000947250	1.13704×10^{-6}
10	0.000948387	0.000948387	0.000948387	0.000946020	2.36678×10^{-6}

2. Fundamental theory of OHAM

Consider the partial differential equation of the form

$$\mathcal{L}(\zeta(x, t)) + \mathcal{N}(\zeta(x, t)) + g(x, t) = 0, x \in \Omega \tag{2}$$

$$\mathcal{B}(\zeta, \partial\zeta/\partial t) = 0,$$

where \mathcal{L} is a linear operator and \mathcal{N} is nonlinear operator. \mathcal{B} is boundary operator, $\zeta(x, t)$ is an unknown function, and x and t denote spatial and time variables, respectively, Ω is the problem domain and $g(x, t)$ is a known function.

Using the basic idea of OHAM, the optimal homotopy $\psi(x, t; a): \Omega \times [0, 1] \rightarrow R$ is constructed which satisfies the following condition.

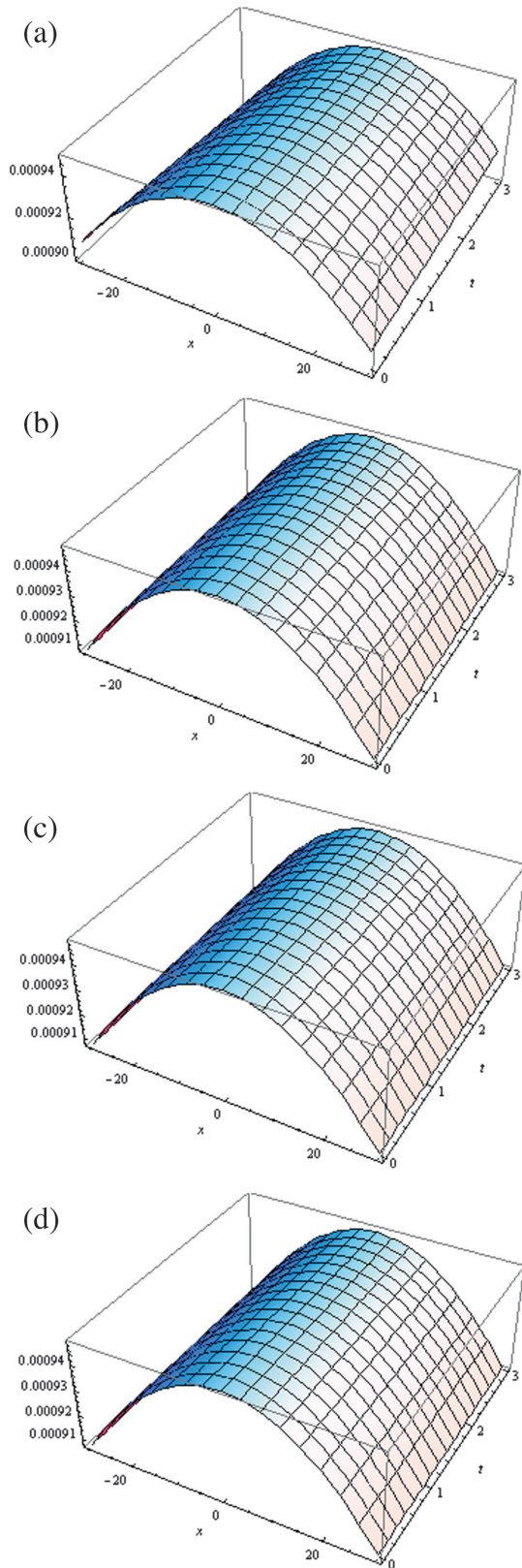


Figure 1 The evolution results for modified Khawara equation for $r = 0.001$, $q = -1$ (1) : (a) Exact (b) OHAM (c) HPM (d) VIM.

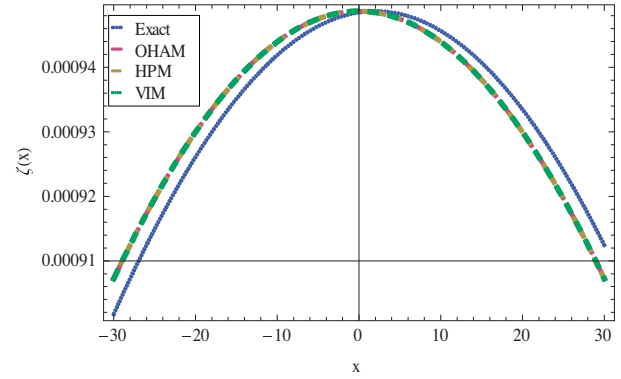


Figure 2 The numerical results for modified Khawara Eq. (1) for $r = 0.001$, $q = -1$ with $t = 2$.

$$(1-a)\{\mathcal{L}(\psi(x,t;a)) + g(x,t)\} = \mathcal{H}(a)\{\mathcal{L}(\psi(x,t;a)) + \mathcal{N}(\psi(x,t;a)) + g(x,t)\}, \quad (3)$$

where $a \in [0,1]$ is an embedding parameter, $\mathcal{H}(a)$ is a nonzero auxiliary function for $a \neq 0$, $\mathcal{H}(0) = 0$. In such a case Eq. (3) is called optimal homotopy equation. Clearly, we have:

$$a = 0 \Rightarrow \mathcal{H}(\psi(x,t;0),0) = \mathcal{L}(\psi(x,t;0)) + g(x,t) = 0, \quad (4)$$

$$a = 1 \Rightarrow \mathcal{H}(\psi(x,t;1),1) = \mathcal{H}(1)\{\mathcal{L}(\psi(x,t;1)) + \mathcal{N}(\psi(x,t;1)) + g(x,t)\} = 0, \quad (5)$$

when $a = 0$ and $a = 1$, then $\psi(x,t;0) = \zeta_0(x,t)$ and $\psi(x,t;1) = \zeta(x,t)$ hold. Thus, as a varies from 0 to 1, the solution $\psi(x,t;a)$ approaches from $\zeta_0(x,t)$ to $\zeta(x,t)$, where $\zeta_0(x,t)$ is obtained from Eq. (3) for $a = 0$:

$$\mathcal{L}(\zeta_0(x,t)) + g(x,t) = 0, \quad \mathcal{B}(\zeta_0, \partial\zeta_0/\partial t) = 0. \quad (6)$$

Next, we choose auxiliary function $\mathcal{H}(a)$ of the following general form

$$\mathcal{H}(a) = aC_1 + a^2C_2 + \dots \quad (7)$$

Here C_1, C_2, \dots are constants to be determined later.

To get an approximate solution, we expand $\psi(x,t;a,ci)$ in Taylor's series about a in the following manner,

$$\psi(x,t;a,C_i) = \zeta_0(x,t) + \sum_{k=1}^{\infty} \zeta_k(x,t;C_i)a^k, \quad i = 1, 2, \dots \quad (8)$$

Substituting Eq. (8) into Eq. (3) and equating the coefficient of like powers of a , we obtain Zeroth order problem, given by Eq. (6), the first and second order problems are given by Eqs. (9) and (10) respectively and the general governing equations for $\zeta_k(x,t)$ are given by Eq. (11):

$$\mathcal{L}(\zeta_1(x,t)) = C_1\mathcal{N}_0(\zeta_0(x,t)), \quad \mathcal{B}(\zeta_1, \partial\zeta_1/\partial t) = 0 \quad (9)$$

$$\mathcal{L}(\zeta_2(x,t)) - \mathcal{L}(\zeta_1(x,t)) = C_2\mathcal{N}_0(\zeta_0(x,t)) + C_1[\mathcal{L}(\zeta_1(x,t)) + \mathcal{N}_1(\zeta_0(x,t), \zeta_1(x,t))], \quad (10)$$

$$\mathcal{B}(\zeta_2, \partial\zeta_2/\partial t) = 0$$

$$\mathcal{L}(\zeta_k(x,t)) - \mathcal{L}(\zeta_{k-1}(x,t)) = C_k\mathcal{N}_0(\zeta_0(x,t)) + \sum_{i=1}^{k-1} C_i[\mathcal{L}(\zeta_{k-i}(x,t)) + \mathcal{N}_{k-i}(\zeta_0(x,t), \zeta_1(x,t), \dots, \zeta_{k-i}(x,t))], \quad (11)$$

$$\mathcal{B}(\zeta_k, \partial\zeta_k/\partial t) = 0, \quad k = 2, 3, \dots,$$

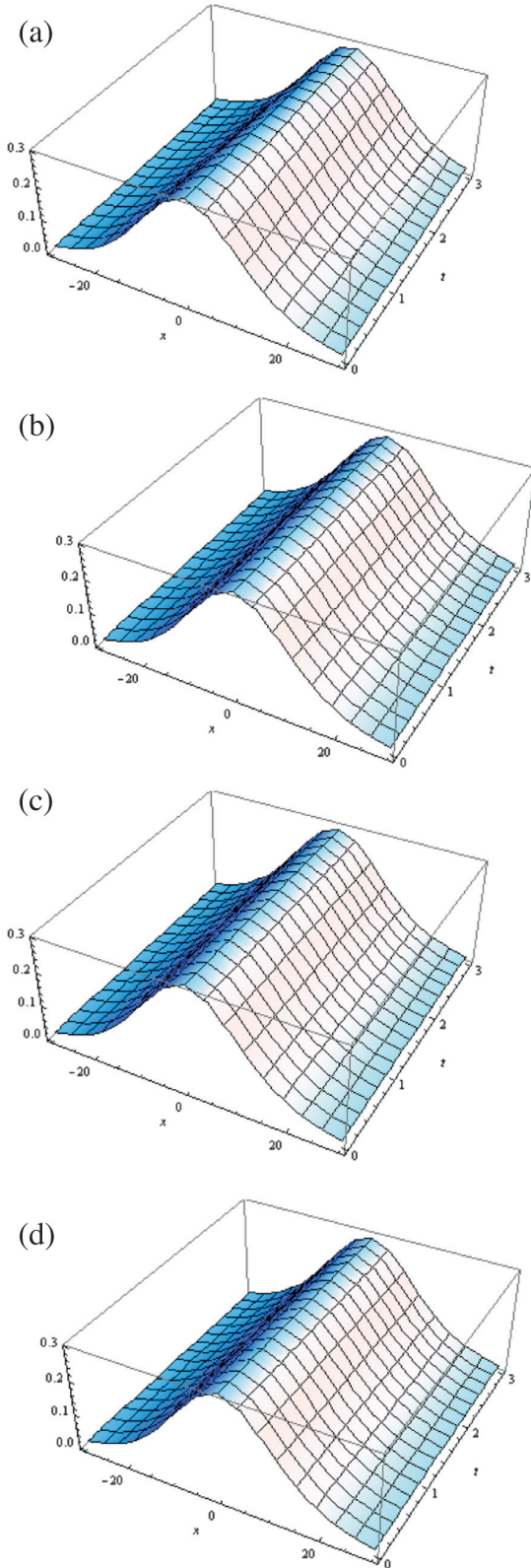


Figure 3 The evolution results for modified Khawara Eq. (1) for $r = 1, q = -10$: (a) Exact (b) OHAM (c) HPM (d) VIM.

where $\mathcal{N}_{k-i}(\zeta_0(x, t), \zeta_1(x, t), \dots, \zeta_{k-i}(x, t))$ is the coefficient of a^{k-i} in the expansion of $\mathcal{N}(\psi(x, t; a))$ about the embedding parameter a .

$$\mathcal{N}(\psi(x, t; a, C_i)) = \mathcal{N}_0(\zeta_0(x, t)) + \sum_{k \geq 1} \mathcal{N}_k(\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_k) a^k. \quad (12)$$

Here ζ_k for $k \geq 0$ are set of linear equations with the linear boundary conditions, which can be easily solved.

The convergence of the series in Eq. (8) depends upon the auxiliary constants C_1, C_2, \dots . If it is convergent at $a = 1$, then:

$$\tilde{\zeta}(x, t; C_i) = \zeta_0(x, t) + \sum_{k \geq 1} \zeta_k(x, t; C_i). \quad (13)$$

Substituting Eq. (13) into Eq. (1), the following expression for residual was obtained:

$$\mathcal{R}(x, t; C_i) = \mathcal{L}(\tilde{\zeta}(x, t; C_i)) + g(x, t) + \mathcal{N}(\tilde{\zeta}(x, t; C_i)). \quad (14)$$

If $\mathcal{R}(x, t; C_i) = 0$, then $\tilde{\zeta}(x, t; C_i)$ will be the exact solution.

For computing the optimal values of auxiliary constants, $C_i, i = 1, 2, \dots, m$, there are many methods available like Galerkin's, Ritz, Least Squares and Collocation method, One can apply the method of Least Squares as under:

$$\mathcal{J}(C_i) = \int_0^t \int_{\Omega} \mathcal{R}^2(x, t, C_i) dx dt, \quad (15)$$

where R is the residual,

$$\mathcal{R}(x, t; C_i) = \mathcal{L}(\tilde{\zeta}(x, t; C_i)) + g(x, t) + \mathcal{N}(\tilde{\zeta}(x, t; C_i)) \quad (16)$$

and

$$\frac{\partial \mathcal{J}}{\partial C_1} = \frac{\partial \mathcal{J}}{\partial C_2} = \dots = \frac{\partial \mathcal{J}}{\partial C_3} = 0. \quad (17)$$

The constants C_i can also be determined by another method as under:

$$\mathcal{R}(h_1; C_i) = \mathcal{R}(h_2; C_i) = \dots = \mathcal{R}(h_m; C_i) = 0, \quad i = 1, 2, \dots, m. \quad (18)$$

at any time t , where $h_i \in \Omega$. The convergence, depends upon constants C_1, C_2, \dots , can be optimally identified and minimized by Eqs. (15)–(18).

3. Application of OHAM to modified Kawahara equation

Consider Eq. (1) with initial condition given by

$$\zeta(x, 0) = \frac{3r}{\sqrt{-10q}} \operatorname{sech}^2(kx). \quad k = \frac{1}{2} \sqrt{\frac{-r}{5q}}, \quad (19)$$

where k is a constant.

Zeroth order problem

In this case $\mathcal{L}(x, t) = \frac{\partial \zeta_0(x, t)}{\partial t}, \quad g(x, t) = 0,$

$$\frac{\partial \zeta_0(x, t)}{\partial t} = 0, \quad \zeta_0(x, 0) = \frac{3r}{\sqrt{-10q}} \operatorname{sech}^2[k(x)]. \quad (20)$$

Its solution is given as under

$$\zeta_0(x, t) = \frac{3r}{\sqrt{-10q}} \operatorname{sech}^2[k(x)]. \quad (21)$$

First order problem

$$\begin{aligned} & -\frac{\partial \zeta_0(x, t)}{\partial t} - C_1 \frac{\partial \zeta_0(x, t)}{\partial t} + \frac{\partial \zeta_1(x, t)}{\partial t} - C_1 \zeta_0^2(x, t) \frac{\partial \zeta_0(x, t)}{\partial x} \\ & - r C_1 \frac{\partial^3 \zeta_0(x, t)}{\partial x^3} - q C_1 \frac{\partial^5 \zeta_0(x, t)}{\partial x^5} = 0, \quad \zeta_1(x, 0) = 0. \end{aligned} \quad (22)$$

Its solution is as follows

$$\zeta_1(x, t, C_1) = -\frac{3t}{50q\sqrt{-q}} \left[\begin{array}{l} -80\sqrt{10}k^3 C_1 q r^2 \operatorname{sech}^2(kx) \tanh(kx) + 1360\sqrt{10}k^5 C_1 q^2 r \operatorname{sech}^6(kx) \tanh(kx) \\ -9\sqrt{10}k C_1 r^3 \operatorname{sech}^6(kx) \tanh(kx) + 40\sqrt{10}k^3 C_1 q r^2 \operatorname{sech}^2(kx) \tanh^2(kx) - \\ 2080\sqrt{10}k^5 C_1 q^2 r \operatorname{sech}^4(kx) \tanh^2(kx) + 160\sqrt{10}k^5 C_1 q^2 r \operatorname{sech}^2(kx) \tanh^5(kx) \end{array} \right]. \quad (23)$$

Second order problem

$$\begin{aligned} -C_2 \frac{\partial \zeta_0}{\partial t} - (1+C_1) \frac{\partial \zeta_1}{\partial t} + \frac{\partial \zeta_2}{\partial t} - C_2 \zeta_0^2 \frac{\partial \zeta_0}{\partial x} - 2C_1 \zeta_0 \zeta_1 \frac{\partial \zeta_0}{\partial x} - C_1 \zeta_0^2 \frac{\partial \zeta_1}{\partial x} \\ - r C_2 \frac{\partial^3 \zeta_0}{\partial x^3} - r C_1 \frac{\partial^3 \zeta_1}{\partial x^3} - q C_2 \frac{\partial^5 \zeta_0}{\partial x^5} - q C_1 \frac{\partial^5 \zeta_1}{\partial x^5} = 0, \quad \zeta_1(x, 0) = 0. \end{aligned} \quad (24)$$

Its solution is

$$\zeta_2(x, t, C_1, C_2) = \frac{3krt \operatorname{sech}^{12}(kx)}{1600\sqrt{10}q^2\sqrt{-q}} \left[\begin{array}{l} -787200k^4 q^3 C_1 \cosh^5(kx) \sinh(kx) + 14400k^2 q^2 r C_1 \cosh^5(kx) \sinh(kx) \\ +2880qr^2 C_1 \cosh^5(kx) \sinh(kx) + 358400k^4 q^3 C_1 \cosh^5(kx) \cosh(2kx) \\ \sinh(kx) + 12800k^2 q^2 r C_1 \cosh^5(kx) \cosh(2kx) \sinh(kx) - 6400k^4 q^3 C_1 \\ \cosh(4kx) \sinh(kx) - 1600k^2 q^2 r C_1 \cosh^5(kx) \cosh(2kx) \sinh(kx) - \\ 12579398400k^9 q^4 t C_1^2 + 54364800k^7 q^3 r t C_1^2 + 35198160k^5 q^2 r^2 t C_1^2 - \\ 103680k^3 q r^3 t C_1^2 - 7776kr^4 t C_1^2 + 15580982400k^9 q^4 t C_1^2 \cosh(2kx) \\ -4536000k^7 q^3 r t C_1^2 \cosh(2kx) - 41966520k^5 q^2 r^2 t C_1^2 \cosh(2kx) - \\ 9360k^3 q t r^3 C_1^2 \cosh(2kx) + 6480kr^4 t C_1^2 \cosh(2kx) - 3525580800k^9 q^4 \\ t C_1^2 \cosh(4kx) - 47616000k^7 q^3 r t C_1^2 \cosh(4kx) + 8100480k^5 q^2 r^2 t C_1^2 \\ \cosh(4kx) + 83520k^3 q r t C_1^2 \cosh(6kx) + 244219200k^9 q^4 t C_1^2 \cosh(6kx) \\ +10884000k^7 q^3 t r^3 C_1^2 \cosh(6kx) - 282540k^5 q^2 t r^2 C_1^2 \cosh(6kx) - \\ 10800k^3 q t r^3 C_1^2 \cosh(6kx) - 3257600k^9 q^4 t C_1^2 \cosh(8kx) - 400000k^7 \\ q^3 t r^3 C_1^2 \cosh(8kx) - 11600k^5 q^2 t r^2 C_1^2 \cosh(8kx) + 1600k^9 q^4 t C_1^2 \\ \cosh(10kx) + 800k^7 q^3 t r C_1^2 \cosh(10kx) + 100k^5 q^2 t r^2 C_1^2 \cosh(10kx) - \\ 100200k^4 q^3 C_1^2 \sinh(2kx) + 3150k^2 q^2 C_1^2 r \sinh(2kx) + 450qr^2 C_1^2 \\ \sinh(2kx) - 64800k^4 q^3 C_1^2 \sinh(4kx) + 3000k^2 q^2 C_1^2 r \sinh(4kx) + \\ 360qC_1^2 r^2 \sinh(4kx) - 2700k^4 q^3 C_1^2 \sinh(6kx) + 1125k^2 q^2 C_1^2 r \\ \sinh(6kx) + 90qC_1^2 r^2 \sinh(6kx) + 5200k^4 q^3 C_1^2 \sinh(8kx) + 100k^2 q^2 \\ C_1^2 r \sinh(8kx) - 100k^4 q^3 C_1^2 \sinh(10kx) - 25k^2 q^2 C_1^2 r \sinh(10kx) - \\ 787200k^4 q^3 C_2 \cosh^5(kx) \sinh(kx) + 14400k^2 q^2 r C_2 \cosh^5(kx) \sinh(kx) \\ +2880qr^2 C_2 \cosh^5(kx) \sinh(kx) + 358400k^4 q^3 C_2 \cosh^5(kx) \cosh(2kx) \\ \sinh(kx) + 12800k^2 q^2 r C_2 \cosh^5(kx) \cosh(2kx) \sinh(kx) - 6400k^4 q^3 C_2 \\ \cosh^5(kx) \cosh(4kx) \sinh(kx) - 1600k^2 q^2 r C_2 \cosh^5(kx) \cosh(4kx) \sinh(kx) \end{array} \right]. \quad (25)$$

Adding Eqs. (21), (23) and (25), we obtained

$$\zeta(x, t) = \zeta_0(x, t) + \zeta_1(x, t, C_1) + \zeta_2(x, t, C_1, C_2). \quad (26)$$

The optimal constants are calculated by using collocation method and its values are

$$C_1 = -0.999996585843794, \quad C_2 = 3.6096969918336 \times 10^{-11}$$

The optimal solution of the modified Kawahara equation is given by

$$\zeta(x, t) = \frac{3r \operatorname{sech}^2(kx)}{\sqrt{-10q}} - \frac{3t}{50\sqrt{-4q}} \left[\begin{aligned} &252.981k^3qr^2 \operatorname{sech}^4(kx) \tanh(kx) - 4300.68k^5q^2r \operatorname{sech}^6(kx) \tanh(kx) \\ &+ 28.4604kr^3 \operatorname{sech}^6(kx) \tanh(kx) - 126.491k^3qr^2 \operatorname{sech}^2(kx) \tanh^3(kx) \end{aligned} \right]$$

$$+ \frac{3krt \operatorname{sech}^2(kx)}{1600\sqrt{10}\sqrt{-4q^2}} \left[\begin{aligned} &-1.25793 \times 10^{10}k^9q^4t + 5.43644 \times 10^{10}k^7q^3rt + 3.51979 \times 10^7k^5q^2r^2t - 103679k^3qr^3t \\ &-7775.95kr^4t + 1.55809 \times 10^{10}k^9q^4t \cosh(2kx) - 4.53597 \times 10^6k^7q^2rt \cosh(2kx) - \\ &4.19662 \times 10^7k^5q^2r^2t \cosh(2kx) - 9359.94k^2qr^3t \cosh(2kx) + 6479.79kr^4t \\ &\cosh(2kx) - 3.52556 \times 10^9k^9q^4t \cosh(4kx) - 4.76157 \times 10^7k^7q^3rt \cosh(4kx) + \\ &8.10042 \times 10^6k^5q^2r^2t \cosh(4kx) - 3.25758 \times 10^6k^9q^4t \cosh(8kx) - 399997k^7q^3 \\ &rt \cosh(8kx) - 11599.9k^5q^2r^2t \cosh(8kx) + 1599.99k^9q^4t \cosh(10kx) + 799.995k^7 \\ &q^3rt \cosh(10kx) + 99.9993k^5q^2r^2t \cosh(10kx) + 787197k^4q^3 \cosh^5(kx) \sinh(kx) - \\ &14400k^2q^2r \cosh^5(kx) \sinh(kx) - 2879.99r^2 \cosh^5(kx) \sinh(kx) - 358399k^4q^3 \\ &\cosh^5(kx) \cosh(2kx) \sinh(kx) - 12800k^2q^2r \cosh^5(kx) \cosh(2kx) \sinh(kx) + 6399.98 \\ &k^4q^3 \cosh^5(kx) \cosh(4kx) \sinh(kx) + 1599.99k^2q^2r \cosh^5(kx) \cosh(4kx) \sinh(kx) - \\ &100199k^4q^3 \sinh(2kx) + 3149.98k^2q^2r \sinh(2kx) + 449.997r^2 \sinh(2kx) - 64799.6 \\ &k^4q^3 \sinh(4kx) + 2999.98k^2q^2r \sinh(4kx) + 359.998qr^2 \sinh(4kx) - 2699.98k^4q^3 \\ &\sinh(6kx) + 3149.98k^2q^2r \sinh(2kx) + 449.997r^2q \sinh(2kx) - 64799.6k^4q^3 \sinh(4kx) \\ &+ 2999.98k^2q^2r \sinh(4kx) + 359.998r^2q \sinh(4kx) - 2699.98k^4q^3 \sinh(6kx) + \\ &1124.99k^2q^2r \sinh(6kx) + 89.9994qr^2 \sinh(6kx) + 5199.96k^4q^3 \sinh(8kx) + 99.9993 \\ &k^2q^2r \sinh(8kx) - 99.9993k^4q^3 \sinh(10kx) - 24.9998k^2q^2r \sinh(10kx) \end{aligned} \right] \quad (27)$$

The VIM results taken from Jin (2009), for Eq. (1) are given by

$$\zeta(x, t)\text{-VIM} = \left[\begin{aligned} &\frac{3r}{\sqrt{-10q}} \operatorname{sech}^2(kx) - \frac{27r^3kt}{5q\sqrt{-10q}} \operatorname{sech}^6(kx) \tanh(kx) - \frac{24r^2k^2t}{\sqrt{-10q}} \operatorname{sech}^2(kx) \tanh(kx) \\ &(2 - 3 \tanh^2(kx)) - \frac{qrk^5t}{\sqrt{-10q}} \operatorname{sech}^2(kx) \tanh(kx) (-2160 \tanh^4(kx) + 2880 \tanh^2(kx) - 816) \end{aligned} \right] \quad (28)$$

The HPM results taken from Jin (2009), for Eq. (1) are given by

$$\zeta(x, t)\text{-HPM} = \left[\begin{aligned} &\frac{3r}{\sqrt{-10q}} \operatorname{sech}^2(kx) - \frac{27r^3kt}{5q\sqrt{-10q}} \operatorname{sech}^6(kx) \tanh(kx) - \frac{24r^2k^2t}{\sqrt{-10q}} \operatorname{sech}^2(kx) \tanh(kx) \\ &(2 - 3 \tanh^2(kx)) - \frac{qrk^5t}{\sqrt{-10q}} \operatorname{sech}^2(kx) \tanh(kx) (-2160 \tanh^4(kx) + 2880 \tanh^2(kx) - 816) \end{aligned} \right] \quad (29)$$

The exact solution of Eq. (1) is given by Sirendaoreji (2004)

$$\zeta(x, t) = \frac{3r}{\sqrt{-10q}} \operatorname{sech}^2[k(x - ct)], \quad (30)$$

where $c = \frac{25q - 4r^2}{25q}$ is a constant.

4. Results and discussion

In Section 2 the fundamental mathematical theory of OHAM is presented which provides highly accurate solutions for the problems demonstrated in Section 3. We have used Mathematica 7 for most of our computational work. In Tables 1–5, we have presented the comparison of VIM, HPM and Exact solutions to OHAM solution and obtained the absolute error of OHAM corresponding to Exact solution for different values of $x = -5.0, -2.5, 0.0, 2.5, 5$. In Fig. 1, 3D plots of VIM,

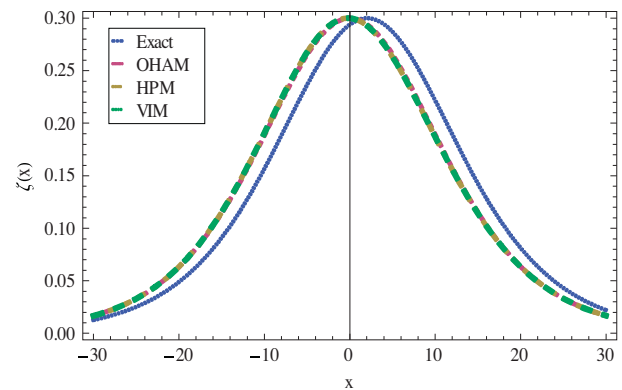


Figure 4 The numerical results for modified Khawara Eq. (1) for $r = 1, q = -10$ with $t = 2$.

HPM, Exact and OHAM solutions are compared and in Fig. 2, 2D plots for VIM, HPM, Exact and OHAM solutions are compared for $p = 0.001$, $q = 1$, $k = 0.00707107$, $c = 1$ at $t = 2$. While in Figs. 3 and 4 the 3D and 2D plots are given for VIM, HPM, Exact and OHAM solutions for $p = 1$, $q = 10$, $k = 0.0707107$, $c = 1.016$ at $t = 2$. It is clear from Figs. 1–4 and Tables 1–5 that OHAM solution is very nearly identical to VIM, HPM and Exact solutions.

5. Conclusion

In this paper, the OHAM has been successfully implemented for the approximate solution of the modified Kawahara equation. The results of these methods have been presented and we conclude that OHAM is very effective, simple, fast convergent and is independent of the assumption of the unrealistic small parameters. The results obtained by OHAM are very consistent in comparison with VIM, HPM, and Exact solutions.

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