Journal of the Association of Arab Universities for Basic and Applied Sciences (2016) **19**, 44–51

University of Bahrain
Journal of the Association of Arab Universities for Basic and Applied Sciences
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حلول مماثلة ذاتية لبعض معادلات التطور اللاخطي: معادلات Kdv وmKdv و برجرز

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الملخص:

يقدم في هذا البحث طريقة لحل بعض معادلات التطور اللاخطي، والتي تشمل الحلول المماثلة ذاتيا، وتعتمد على تقليل التشابه في متغيرات الزمن والفراغ، ومن ثم الوصول إلى تقليل التماثل لهذه المعادلات.

الحلول المماثلة ذاتيا لمعادلات KdV، mKdV، مرتبطة بدوال بسل، إيري الخاصة، بينما حلول معادلة برجرز مرتبطة بدوال الخطأ، هيرميت الخاصة. هذه الحلول المستنتجة بالبحث تمثل نوعا جديدا من الموجات السولوتنية، الصدمية، الدورية، يمكن تطبيق الطريقة المقدمة لمعادلات تطور لا خطية أخرى في مجال الفيزياء الرياضية.

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ORIGINAL ARTICLE

Self-similar solutions for some nonlinear evolution equations: KdV, mKdV and Burgers equations

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Received 4 January 2014; revised 20 May 2014; accepted 12 June 2014
Available online 15 July 2014

KEYWORDS

Self-similar solution;
Symbolic computations;
Exact solutions;
KdV, mKdV and Burgers equations

Abstract A method for solving three types of nonlinear evolution equations namely KdV, modified KdV and Burgers equations, with self-similar solutions is presented. The method employs ideas from symmetry reduction to space and time variables and similarity reductions for nonlinear evolution equations are performed. The obtained self-similar solutions of KdV and mKdV equations are related to Bessel and Airy functions whereas those of Burgers equation are related to the error and Hermite functions. These solutions appear as new types of solitary, shock and periodic waves. Also, the method can be applied to other nonlinear evolution equations in mathematical physics.
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1. Introduction

Investigating exact solutions of nonlinear evolution equations plays an important role in mathematical physics. The complex phenomena in various fields of sciences, especially in physics such as fluid dynamics, plasma, optical fibers, solid state physics, nonlinear optics and so on can be characterized in terms of the Korteweg–de Vries (KdV), modified Korteweg–de Vries (mKdV) and Burgers equations (Whitham, 1974; Davidson, 1972; Agarwal, 1995; Toda, 1981). Therefore, a great deal of research work has been invested during the past decades for the study of the nonlinear evolution equations. The main goals of these studies were directed towards the analytical and numerical solutions. Several different approaches, such as inverse scattering method (Ablowitz and Clarkson, 1991), the tanh function method (Malfiet, 1992; Evans and Raslan, 2005), extended tanh method (Fan, 2000; Elwakil et al., 2002; Yusufoglu and Bekir, 2007), the homogenous balance method (Wang et al., 1996; Jafari et al., 2014), the exp-function method (He and Abdou, 2007; Matinfar et al., 2014), the extended F-expansion method (Abdou, 2007; Zhao, 2013), the Jacobi elliptic function expansion method (Abdou and Elhanbaly, 2007; Gepreel and Alzaidy, 2012), homotopy analysis method (Abdou, 2009; Ghanbari, 2014) and many others were employed to obtain more insights through the physical behavior of these solutions.

Symmetry reduction has long been recognized as a useful tool for finding and analyzing self-similar solutions of partial differential equations (PDEs) (Olver, 1986; Ovsiannikov, 1982; Bernoff and Witelski, 2010; McCoy, 2011). The usual

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Peer review under responsibility of University of Bahrain.

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http://dx.doi.org/10.1016/j.jaubas.2014.06.007

approach involves assuming a particular form of the solution and introducing a change of variables that simplifies the governing equations. Self-similar solutions provide some of the greatest simplifications to one-dimensional flows. Self-similarity allows the reduction of the partial differential equations, which contain two independent variables (space and time), into a set of ordinary differential equations (ODEs), where the single independent variable is a combination of space and time. The ODEs are then relatively easy to be solved numerically or even analytically in some cases (Changxing and Bo, 2006; Perez-Garcia, 2004; Huang, 2010; Kudryashov, 2009; Kraenkel et al., 2011; Ngamsaad and Khompurngson, 2012).

The asymptotic behavior of one-dimensional flows in a variety of circumstances is described, typically far away from the initial conditions provided that the boundary conditions contain no spatial scale (Barenblatt, 1996; Sari, 2006).

In this article, we provide a self-similar technique for finding analytical solutions for KdV, mKdV and Burgers equations.

The present paper proceeds as follows: in Section 2, an algorithm describing the self-similar scaling is presented. In Section 3, we introduce the novel technique method used to solve the variable coefficient non-linear equation in one dimension resulting from the self-similar transformation. In the next Section 4, we apply a computerized approach for finding the exact solutions of the KdV, mKdV and Burgers equations. Finally, some discussions and conclusions are given in Section 5.

2. Self similar transformation

A self-similar solution is a solution of the form:

$$w(x,t) = t^{\alpha}u(\xi), \quad \xi = xt^{\beta}, \tag{1}$$

$$H(x,t,w_x,w_t,w_{xx},w_{xt},w_{tt},\ldots) = 0. \tag{2}$$

Self-similar solutions exist if the scaling of the independent and dependent variables follows

$$t = c\tau, \quad x = c^k\varsigma \quad \text{and} \quad \omega(\varsigma) = c^m w(x,t), \tag{3}$$

where $c \neq 0$ is an arbitrary constant.

Substituting (3) into (2), it turns into the same equation in the new variables,

$$H(\varsigma,\tau,w_\varsigma,w_\tau,w_{\varsigma\varsigma},w_{\varsigma\tau},w_{\tau\tau},\ldots) = 0. \tag{4}$$

If a pair of the parameters k and m in (3) has been found such that (4) holds true, there is a self-similar solution of the form (1), where $k = \beta$ and $m = \alpha$.

These relations follow from the condition that the scaling transformation (3) must preserve the form of the variables (1):

$$w(x,t) = t^{\alpha}u(\xi), \quad \xi = xt^{\beta} \Rightarrow \omega(\varsigma,\tau) = \tau^{\alpha}U(\eta), \quad \eta = \varsigma\tau^{\beta}. \tag{5}$$

3. The computerized technique

It is found that the equation resulting from self-similar transformation is a variable coefficient ordinary nonlinear differential equation.

$$H(\xi,u_\xi,u_{\xi\xi},\ldots) = 0. \tag{6}$$

The computerized technique is organized as follows:

Step (1): Using the expansion

$$u(\xi) = \sum_{i=0}^{n} a_i(\xi)\varphi(\xi)^i, \tag{7}$$

where n is an integer to be determined by balancing the highest order derivative term with the highest nonlinear term. The coefficients $a_i(\xi)$, $(i = 0,\ldots,n)$ and the function $\varphi(\xi)$ are all differentiable functions and $\varphi(\xi)$ is the solution of the generalized Riccati equation

$$\frac{d\varphi(\xi)}{d\xi} = c_0(\xi) + c_1(\xi)\varphi(\xi) + c_2(\xi)\varphi(\xi)^2. \tag{8}$$

where c_0, c_1 and c_2 are arbitrary functions for ξ to be determined later.

Step (2): Substituting (7) into (6) and making use of (8) then set the coefficients of $\varphi(\xi)^i$, $(i = 0,\ldots,n)$ equal to zero, we get a set of differential equations with regard to the differential functions $a_i(\xi)$, $(i = 0,\ldots,n)$ and c_j (ξ), $(j = 0,1,2)$.

Step (3): Solving the over-determined ordinary differential equations, using symbolic computation packages like Mathematica or Maple, will lead to the explicit expressions for $a_i(\xi)$ and c_j (ξ) or the constraints among them.

Step (4): Taking the coefficients $c_j(\xi)$, $(j = 0,1,2)$ and substituting into (8) derives the corresponding general solution $\varphi(\xi)$ of (8).

Step (5): Substituting the general solution $\varphi(\xi)$ and the coefficients $a_i(\xi)$, $(i = 0,\ldots,n)$ into expansion (7) gives the exact solution of (6).

4. Self-similar solution of some nonlinear evolution equations

4.1. KdV equation

Let us consider the KdV equation of the form:

$$\frac{\partial}{\partial t}w(x,t) + Aw(x,t)\frac{\partial}{\partial x}w(x,t) + B\frac{\partial^3}{\partial x^3}w(x,t) = 0, \tag{9}$$

where $w(x,t)$ is the distribution function in terms of the space x and the time t, A is the nonlinear coefficient and B is the dispersion coefficient.

A self-similar transformation to this equation can be taken of the form

$$w(x,t) = t^{\alpha}u(\xi), \quad \xi = xt^{\beta}. \tag{10a}$$

Substituting this assumption into (9) and making the resultant equation invariant lead to yield $\alpha = -2/3$ and $\beta = -1/3$. Therefore, the self-similar transformation for the KdV equation (9) is

$$w(x,t) = t^{-2/3}u(\xi), \quad \xi = xt^{-1/3}. \tag{10b}$$

Substituting (10b) into (9), one gets the variable coefficients nonlinear ordinary differential equation:

$$3B\frac{d^3}{d\xi^3}u(\xi) - \xi\frac{d}{d\xi}u(\xi) + 3Au(\xi)\frac{d}{d\xi}u(\xi) - 2u(\xi) = 0. \tag{11}$$

In order to proceed further, we employ the computerized technique presented in Section 3. Balancing the highest derivative term with nonlinear term in (11) leads to $n = 2$. Hence we may assume that

$$u(\xi) = a_0(\xi) + a_1(\xi)\varphi(\xi) + a_2(\xi)\varphi(\xi)^2. \tag{12}$$

Substituting (12) and (8) into (11) and setting the coefficients of different powers of $\varphi(\xi)$ equal to zero, we have a set of over-determined ordinary differential equations for $a_i(\xi)$, (i = 0,1,2) and $c_j(\xi)$, (j = 0,1,2). With the aid of the Maple or Mathematica package, the obtained system of differential equations is solved and yields:

$$a_0(\xi) = -[48B\xi c_2(\xi)^2 + A^2 a_1(\xi)^2]/[48ABc_2(\xi)^2], \tag{13a}$$
$$a_1(\xi) = -12(B/A)\{c_1(\xi)c_2(\xi) + [dc_2(\xi)/d\xi]\}, \tag{13b}$$
$$a_2(\xi) = -12(B/A)c_2(\xi)^2, \tag{13c}$$
$$c_0(\xi) = \{24ABc_2(\xi)[da_1(\xi)/d\xi] + A^2 a_1(\xi)^2 + 96\xi Bc_2(\xi) - 24ABa_1(\xi)[dc_2(\xi)/d\xi]\}/[576B^2 c_2(\xi)^3], \tag{13d}$$

with $c_1(\xi)$ and $c_2(\xi) \neq 0$ are arbitrary functions in ξ.

As an application to this solution, we will study two special cases:

4.1.1. Special case (1)

Let $A = -6$, $B = 1$, $c_1(\xi) = 0$ and $c_2(\xi) = 1$. The remained coefficients become

$$a_0(\xi) = \xi/6, \quad a_1(\xi) = 0, \quad a_2(\xi) = 2, \quad c_0(\xi) = \xi/6. \tag{14}$$

The corresponding solution of the Riccati equation is given as

$$\varphi(\xi) = \frac{\xi^{3/2}\left[C_1 J_{2/3}\left(\sqrt{\frac{2\xi^3}{27}}\right) - C_1 J_{-4/3}\left(\sqrt{\frac{2\xi^3}{27}}\right) - 2J_{-2/3}\left(\sqrt{\frac{2\xi^3}{27}}\right)\right] - \sqrt{6}C_1 J_{-1/3}\left(\sqrt{\frac{2\xi^3}{27}}\right)}{2\sqrt{6}\xi\left[J_{1/3}\left(\sqrt{\frac{2\xi^3}{27}}\right) + C_1 J_{-1/3}\left(\sqrt{\frac{2\xi^3}{27}}\right)\right]}, \tag{15a}$$

where $J_v(.)$ is the Bessel function of the first kind of order v and C_1 is an arbitrary constant. Taking the constant $C_1 = 0$, the Riccati equation solution has the form

$$\varphi(\xi) = -\sqrt{\frac{\xi}{6}}\frac{J_{-2/3}\left(\sqrt{\frac{2\xi^3}{27}}\right)}{J_{1/3}\left(\sqrt{\frac{2\xi^3}{27}}\right)}. \tag{15b}$$

Substituting (15b) into (12) with the aid of (14), we get the solution of the self-similar form of the KdV equation (11) for $C_1 = 0$ as:

$$u(\xi) = \frac{\xi}{3}\left\{\frac{1}{2} + \left[\frac{J_{-2/3}\left(\sqrt{\frac{2}{27}\xi^3}\right)}{J_{1/3}\left(\sqrt{\frac{2}{27}\xi^3}\right)}\right]^2\right\}, \quad \xi = xt^{-1/3}. \tag{16}$$

Therefore, the solution of the KdV equation in terms of the original coordinates (x, t) for this case is given as

$$w(x,t) = \frac{x}{3t}\left\{\frac{1}{2} + \left[\frac{J_{-2/3}\left(\sqrt{\frac{2}{27}\frac{x^3}{t}}\right)}{J_{1/3}\left(\sqrt{\frac{2}{27}\frac{x^3}{t}}\right)}\right]^2\right\}. \tag{17}$$

4.1.2. Special case (2)

Let $A = -6$, $B = 1$, $c_1(\xi) = \sin(\xi)$ and $c_2(\xi) = \operatorname{sech}(\xi)$. The coefficients $a_0(\xi)$, $a_1(\xi)$, $a_2(\xi)$, $c_0(\xi)$ and the solution of the Riccati equation (8), $\varphi(\xi)$, are given as

$$a_0(\xi) = \frac{\xi}{6} + \frac{1}{2}[\sin(\xi) - \tanh(\xi)]^2, \tag{18a}$$
$$a_1(\xi) = 2\operatorname{sech}(\xi)[\sin(\xi) - \tanh(\xi)], \tag{18b}$$
$$a_2(\xi) = 2\operatorname{sech}^2(\xi), \tag{18c}$$
$$c_0(\xi) = \frac{1}{2}\cosh(\xi)\left[\frac{1}{2} + \frac{1}{3}\xi - \cos(\xi) + \frac{1}{2}\sin^2(\xi)\right] - \frac{1}{2}\sin(\xi)\sinh(\xi) + \frac{1}{4}\operatorname{sech}(\xi) \tag{18d}$$

and

$$\varphi(\xi) = \frac{\cosh(\xi)\left[\mathrm{AiryBi}\left(1, -\sqrt[3]{\tfrac{1}{6}\xi}\right) + C_1\mathrm{AiryAi}\left(1, -\sqrt[3]{\tfrac{1}{6}\xi}\right)\right]}{\sqrt[3]{6}\left[\mathrm{AiryBi}\left(-\sqrt[3]{\tfrac{1}{6}\xi}\right) + C_1\mathrm{AiryAi}\left(-\sqrt[3]{\tfrac{1}{6}\xi}\right)\right]} - \frac{1}{2}[\cosh(\xi)\sin(\xi) - \sinh(\xi)], \tag{19a}$$

where C_1 is an arbitrary constant, AiryAi and AiryBi are the Airy wave functions.

For $C_1 = 0$, the solution of Riccati equation is given as

$$\varphi(\xi) = \frac{\cosh(\xi)}{\sqrt[3]{6}}\left[\frac{\mathrm{AiryBi}\left(1, -\sqrt[3]{\tfrac{1}{6}\xi}\right)}{\mathrm{AiryBi}\left(-\sqrt[3]{\tfrac{1}{6}\xi}\right)}\right] - \frac{1}{2}[\cosh(\xi)\sin(\xi) - \sinh(\xi)]. \tag{19b}$$

Substituting (19b) into (12) with the aid of (18) leads to the solution of (11) for $C_1 = 0$ as:

$$u(\xi) = \frac{\xi}{6} + \sqrt[3]{\frac{2}{9}}\left[\frac{\mathrm{AiryBi}\left(1, -\sqrt[3]{\tfrac{1}{6}\xi}\right)}{\mathrm{AiryBi}\left(-\sqrt[3]{\tfrac{1}{6}\xi}\right)}\right]^2, \quad \xi = xt^{-1/3}. \tag{20}$$

Therefore, the exact solution of the KdV equation in this case is given as

$$w(x,t) = \frac{x}{6t} + \sqrt[3]{\frac{2}{9t^2}}\left[\frac{\mathrm{AiryBi}\left(1, -\sqrt[3]{\frac{x^3}{6t}}\right)}{\mathrm{AiryBi}\left(-\sqrt[3]{\frac{x^3}{6t}}\right)}\right]^2. \tag{21}$$

4.2. mKdV equation

Let us consider mKdV equation of the form:

$$\frac{\partial}{\partial t}w(x,t) + Aw(x,t)^2\frac{\partial}{\partial x}w(x,t) + B\frac{\partial^3}{\partial x^3}w(x,t) = 0, \tag{22}$$

where $w(x, t)$ is the distribution function, A and B are the nonlinear coefficient and the dispersion coefficient, respectively.

The self-similar transformation of this equation can be represented as

$$w(x,t) = t^{\alpha}u(\xi), \quad \xi = xt^{\beta}. \tag{23a}$$

Substituting this assumption into (22) and making the resultant equation invariant lead to the exponents of the self-similar

transformation as $\alpha = -1/3$ and $\beta = -1/3$. Therefore, the self-similar transformation for the mKdV equation (22) is given as

$$w(x,t) = t^{-1/3}u(\xi), \quad \xi = xt^{-1/3}. \tag{23b}$$

Inserting (23b) into (22) gives a nonlinear ordinary differential equation with variable coefficients of the form:

$$3B\frac{d^3}{d\xi^3}u(\xi) - \xi\frac{d}{d\xi}u(\xi) + 3Au^2(\xi)\frac{d}{d\xi}u(\xi) - u(\xi) = 0. \tag{24}$$

The balancing of the highest derivative term with the nonlinear term in (24) leads to $n = 1$. Hence we may assume the solution of (24) as

$$u(\xi) = a_0(\xi) + a_1(\xi)\varphi(\xi). \tag{25}$$

Substituting (25) and (8) into (24) and setting the coefficients of $\varphi(\xi)^i$, $(i = 0,1)$ equal to zero lead to a set of over-determined ordinary differential equations for the coefficients $a_i(\xi)$, $(i = 0,1)$ and $c_j(\xi)$, $(j = 0,1,2)$. The solution of the obtained system of the differential equations, using the Maple or Mathematica, leads to:

$$a_0(\xi) = \pm\frac{\sqrt{-6AB}}{2A}\left[c_1(\xi) + \frac{1}{c_2(\xi)}\frac{dc_2(\xi)}{d\xi}\right], \tag{26a}$$

$$a_1(\xi) = \pm\sqrt{-6B/A}c_2(\xi), \tag{26b}$$

$$c_0(\xi) = \left\{\xi - Aa_0(\xi)^2 \pm \sqrt{-6AB}[da_0(\xi)/d\xi]\right\}/[6Bc_2(\xi)], \tag{26c}$$

with $c_1(\xi)$ and $c_2(\xi) \neq 0$ are arbitrary functions in ξ.

As application to this solution, we will study the following special case:

4.2.1. Special case

Let $A = -6$, $B = 1$, $c_1(\xi) = 0$ and $c_2(\xi) = 1$. The remained coefficients become

$$c_0(\xi) = \xi/6, \quad a_0(\xi) = 0, \quad a_1(\xi) = 1, \tag{27}$$

and the corresponding solution of the Riccati equation is given as

$$\varphi(\xi) = \left[C_1\mathrm{AiryAi}\left(1, -\sqrt[3]{\frac{\xi^3}{6}}\right) + \mathrm{AiryBi}\left(1, -\sqrt[3]{\frac{\xi^3}{6}}\right)\right] \Big/ \left\{\sqrt[3]{6}\left[C_1\mathrm{AiryAi}\left(-\sqrt[3]{\frac{\xi^3}{6}}\right) + \mathrm{AiryBi}\left(-\sqrt[3]{\frac{\xi^3}{6}}\right)\right]\right\}, \tag{28a}$$

where $J_v(.)$ is the Bessel function of the first kind of order v and C_1 is an arbitrary constant.

The solution of the Riccati equation (8) for the case of $C_1 = 0$ has the form

$$\varphi(\xi) = \left[\mathrm{AiryBi}\left(1, -\sqrt[3]{\frac{\xi^3}{6}}\right)\right] \Big/ \left\{\sqrt[3]{6}\left[\mathrm{AiryBi}\left(-\sqrt[3]{\frac{\xi^3}{6}}\right)\right]\right\}. \tag{28b}$$

Substituting (28b) into (25) with the aid of (27), we get the self-similar solution of the mKdV equation (24) for $C_1 = 0$ as:

$$u(\xi) = \left[\mathrm{AiryBi}\left(1, -\sqrt[3]{\frac{\xi^3}{6}}\right)\right] \Big/ \left\{\sqrt[3]{6}\left[\mathrm{AiryBi}\left(-\sqrt[3]{\frac{\xi^3}{6}}\right)\right]\right\}, \quad \xi = xt^{-1/3}. \tag{29}$$

Therefore, the exact solution of the mKdV equation for this case is given as

$$w(x,t) = \left[\mathrm{AiryBi}\left(1, -\sqrt[3]{\frac{x^3}{6t}}\right)\right] \Big/ \left\{\sqrt[3]{6t}\left[\mathrm{AiryBi}\left(-\sqrt[3]{\frac{x^3}{6t}}\right)\right]\right\}. \tag{30}$$

4.3. Burgers equation

The Burgers equation has the form

$$\frac{\partial}{\partial t}w(x,t) + Aw(x,t)\frac{\partial}{\partial x}w(x,t) + B\frac{\partial^2}{\partial x^2}w(x,t) = 0. \tag{31}$$

A self-similar transformation for the Burgers equation is given in the form

$$w(x,t) = t^{\alpha}u(\xi), \quad \xi = xt^{\beta}. \tag{32a}$$

Using this assumption into (31) and making the resulting equation invariant give the transformation exponents as $\alpha = -1/2$ and $\beta = -1/2$. Therefore, the self-similar solution of this Burgers equation has the form

$$w(x,t) = t^{-1/2}u(\xi), \quad \xi = xt^{-1/2}. \tag{32b}$$

Inserting (32b) into (31), the Burgers equation (31) is reduced to the following variable coefficients nonlinear ordinary differential equation:

$$2B\frac{d^2}{d\xi^2}u(\xi) - \xi\frac{d}{d\xi}u(\xi) + 2Au(\xi)\frac{d}{d\xi}u(\xi) - u(\xi) = 0. \tag{33}$$

Balancing the highest derivative term with nonlinear term in (33) leads to $n = 1$. Hence we may assume a solution of (33) as:

$$u(\xi) = a_0(\xi) + a_1(\xi)\varphi(\xi). \tag{34}$$

Substituting (34) into (33), then setting the coefficients of $\varphi(\xi)^i$, $(i = 0,1)$ to zero, we get a set of differential equations with regard to the differentiable functions $a_i(\xi)$, $(i = 0,1)$ and $c_j(\xi)$, $(j = 0,1,2)$. Solving the obtained system of equation leads to the following coefficients:

$$a_0(\xi) = \frac{1}{A}\left[\frac{\xi}{2} - Bc_1(\xi) - \frac{B}{c_2(\xi)}\frac{d}{d\xi}c_2(\xi)\right], \tag{35a}$$

$$a_1(\xi) = -2Bc_2(\xi)/A, \tag{35b}$$

$$c_0(\xi) = -\frac{A}{4B^2c_2(\xi)}\left[\xi a_0(\xi) - Aa_0(\xi)^2 - 2B\frac{d}{d\xi}a_0(\xi)\right], \tag{35c}$$

with $c_1(\xi)$ and $c_2(\xi) \neq 0$ are arbitrary functions in ξ.

Due to the presence of the arbitrary functions, two special cases are in order.

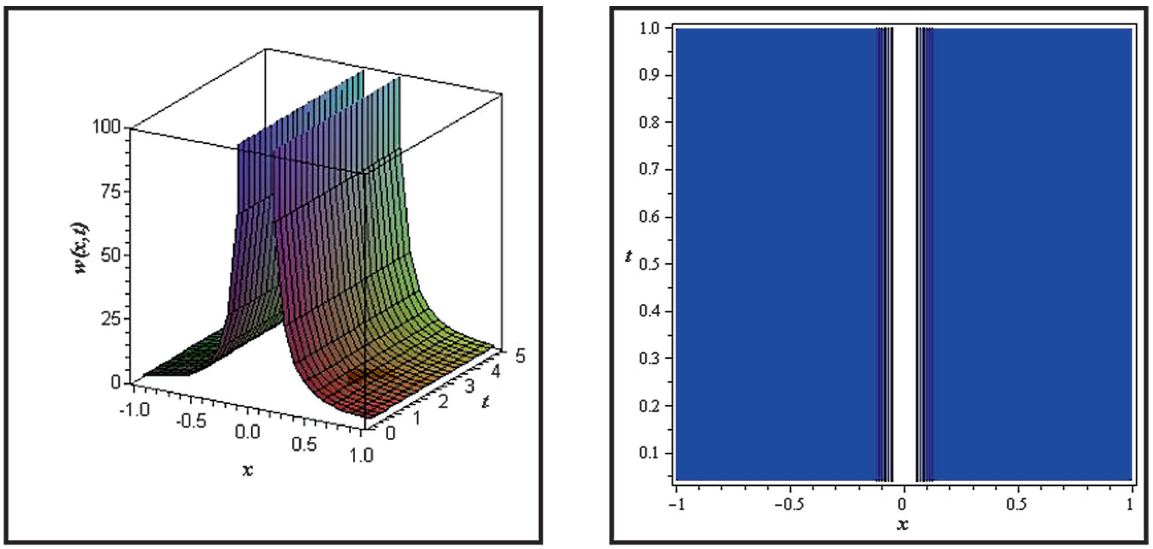
4.3.1. Special case (1)

Let the nonlinear coefficient be $A = -1$ and the dispersion coefficient be $B = -1$ while the arbitrary coefficients of Riccati equation are $c_1(\xi) = 0$ and $c_2(\xi) = 1$. This leads to

$$a_0(\xi) = -\xi/2, \quad a_1(\xi) = -2, \quad c_0(\xi) = -(\xi^2 + 4)/16 \tag{36}$$

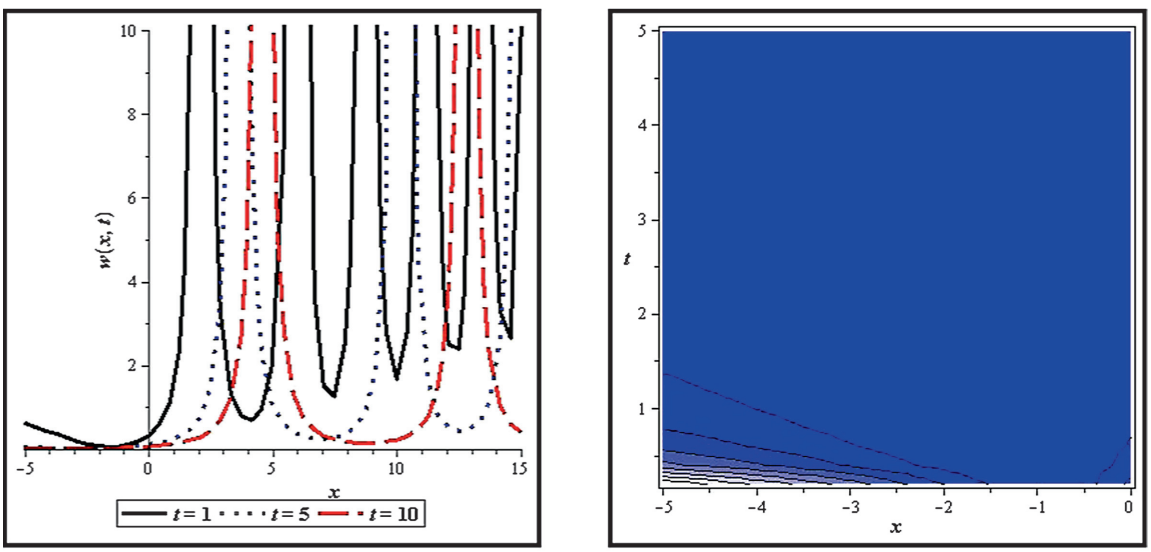
and

$$\varphi(\xi) = -\frac{\xi}{4} + \frac{\exp(-\xi^2/4)}{C_1 - \sqrt{\pi}\,\mathrm{erf}(\xi/2)}, \tag{37a}$$



(a) (b)

Figure 1 The profile and contour plot of explosive solution (17). Light-colored regions correspond to higher values of the soliton amplitude.



(a) (b)

Figure 2 The profile and contour plot of the damped periodic explosive solution (21). Light-colored regions correspond to higher values of the soliton amplitude.

where C_1 is an arbitrary constant and $\mathrm{erf}(\xi/2)$ is the error function. If $C_1 = 0$, the solution of the Riccati equation in this case is

$$\varphi(\xi) = -\frac{\xi}{4} - \frac{\exp(-\xi^2/4)}{\sqrt{\pi}\,\mathrm{erf}(\xi/2)}, \tag{37b}$$

The solution of the nonlinear ordinary differential equation (33) in this case becomes:

$$u(\xi) = [2\exp(-\xi^2/4)]/\left[\sqrt{\pi}\,\mathrm{erf}(\xi/2)\right], \tag{38}$$

And hence, the solution of Burgers equation has the form

$$w(x,t) = \left[2\exp\left(-\frac{x^2}{4t}\right)\right] \Big/ \left[\sqrt{\pi t}\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right]. \tag{39}$$

4.3.2. Special case (2)

Let the nonlinear and the dispersion coefficients are $A = B = -1$ and the coefficients in the Riccati equation (8) are $c_1(\xi) = 0$ and $c_2(\xi) = \mathrm{sech}(\xi)$. The coefficients $a_0(\xi)$, $a_1(\xi)$ and $c_0(\xi)$ and the solution of the Riccati equation, $\varphi(\xi)$, are given as

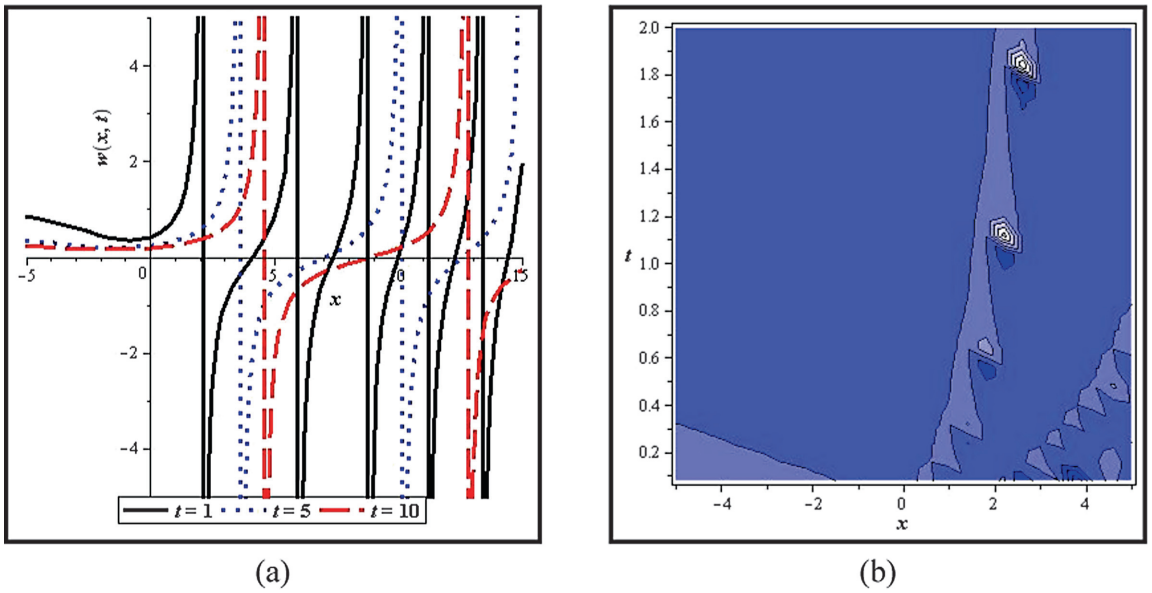


Figure 3 The profile and contour plot of the periodic explosive shock solution (30).

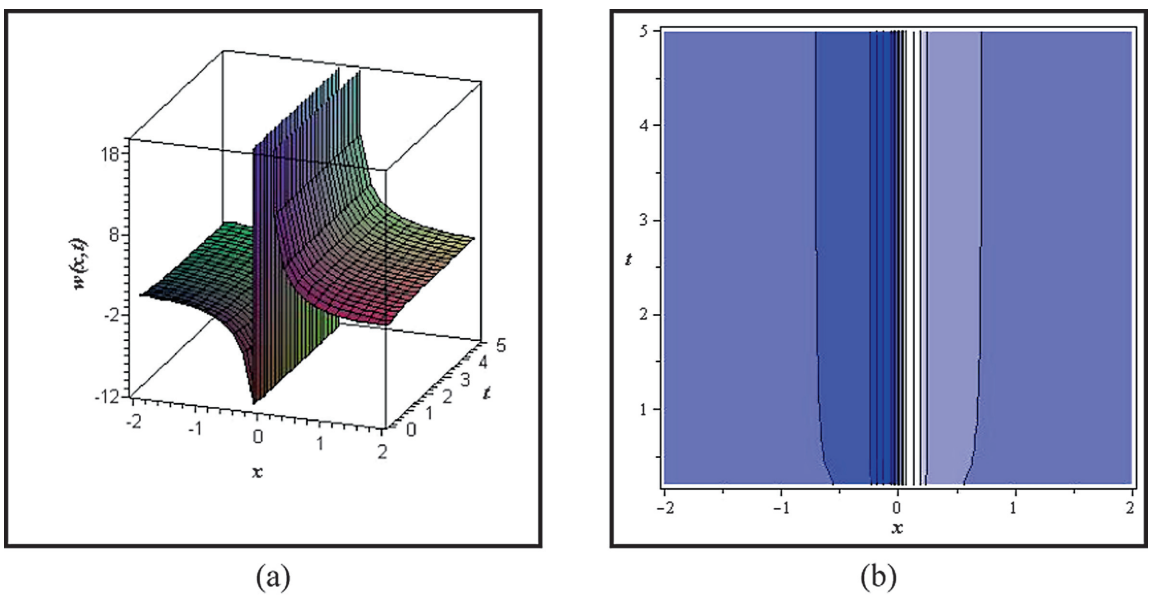


Figure 4 The profile and contour plot of the solution (39).

$$a_0(\xi) = -\xi/2 + \tanh(\xi), \quad a_1(\xi) = -2\text{sech}(\xi),$$
$$c_0(\xi) = [4\text{sech}(\xi) - \xi^2 \cosh(\xi)]/16 \tag{40}$$

and

$$\varphi(\xi) = \{2\sinh(\xi) - \xi\cosh(\xi) + 2C_1\cosh(\xi)/[\exp(\xi^2/4) + C_1 H_{-1}(\xi/2)]\}/4, \tag{41a}$$

where H_n is the Hermite function and C_1 is an arbitrary constant. This Riccati equation solution for $C_1 = 1$ is given in the form

$$\varphi(\xi) = \{2\sinh(\xi) - \xi\cosh(\xi) + 2\cosh(\xi)/[\exp(\xi^2/4) + H_{-1}(\xi/2)]\}/4. \tag{41b}$$

Substituting into (34), the solution of (33) with $C_1 = 1$ becomes:

$$u(\xi) = -1/[\exp(\xi^2/4) + H_{-1}(\xi/2)]. \tag{42}$$

Therefore, the solution of Burgers equation (31) in this case is given by

$$w(x,t) = -1 \Big/ \left\{ \sqrt{t}\left[\exp\left(\frac{x^2}{4t}\right) + H_{-1}\left(\frac{x}{2\sqrt{t}}\right)\right]\right\}. \tag{43}$$

5. Results and discussion

In this paper, based on the computerized symbolic computation, the nonlinear KdV, mKdV and Burgers equations have

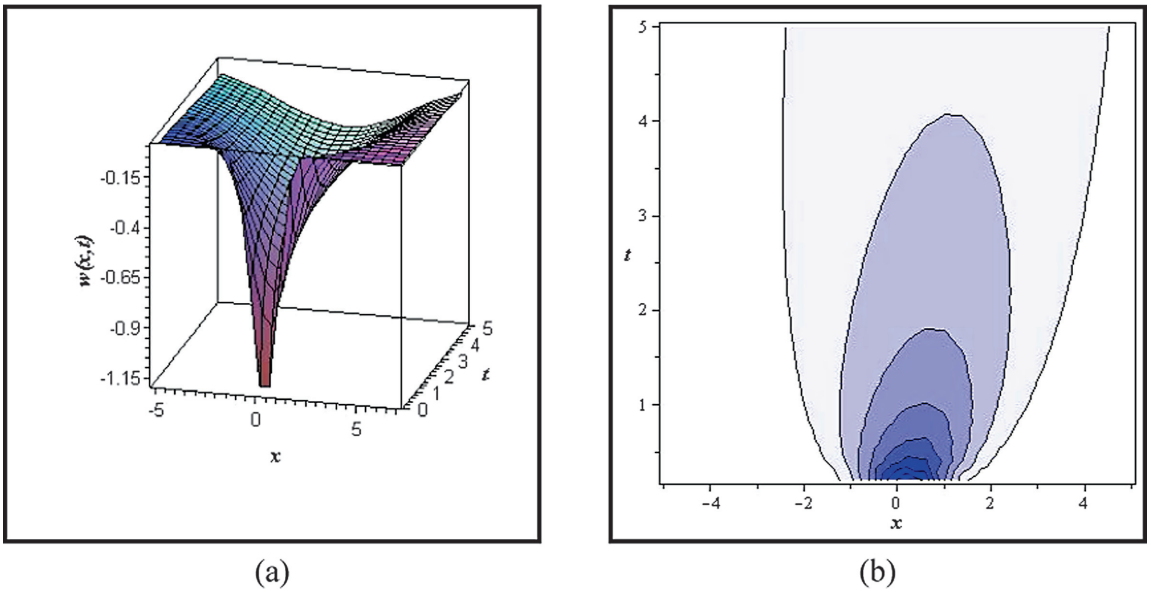


Figure 5 The profile and contour plot of the solution (43). Light-colored regions correspond to higher values of the soliton amplitude.

been investigated. In order to reveal the physical properties, the actual application of the solitons, shock-like solutions often requires an advisable value for the involved parameters and functions. Hence for the picture drawing and qualitative analysis, we choose some values for those parameters and functions in line with the non-uniform backgrounds of the model under investigation. These solutions are presented and discussed through the figures. Solution (17) leads to the propagation of a localized explosive pulse (Sabry et al., 2008), as depicted in Fig. 1. Fig. 2 shows the profile of a periodic blowup explosive soliton solution (21). Eq. (30) is a periodic explosive shock solution as shown in Fig. 3. Rarefactive and compressive explosive shocks in x and t directions for solution (39) are depicted in Fig. 4. For solution (43), Fig. 5 shows that a localized solitary pulse is obtained for large value of t but a localized explosive solitary pulse for a small value of t (Moslem et al., 2009). In summary, it has been found that the self-similar analysis modifies the solutions of KdV, mKdV and Burgers equations significantly and new exact solutions have been obtained. To our knowledge, these solutions have not been reported. It may be important to explain some physical phenomena.

6. Conclusions

In this work, we developed an analytical method for obtaining self-similar solutions for some nonlinear equations like KdV, mKdV and Burgers equations which have a very important role for investigation of many physical phenomena in physics and engineering. Our solutions have different wave structures, such as explosive solitary, shock and periodic waves depending on the arbitrary functions. These solutions may be important to explain some physical phenomena.

Normally, previously obtained solutions of KdV, mKdV and Burgers equations are based on the ansatz of traveling wave transformation that correspond to time and space translation invariance. This leads to that the reduced equations are nonlinear ordinary differential equations with constant coefficients. But, our solutions that are based on self-similar approach correspond to time and space scaling invariance. This leads to that the reduced equations are nonlinear ordinary differential equations with variable coefficients. Therefore, our results are completely different from those obtained previously using traveling wave ansatz. The used mathematical method can be extended to solve either nonlinear integro-differential equations or other classes of differential equations.

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