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ORIGINAL ARTICLE

Stability and boundedness in delay system of differential equations of third order

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Abstract In this paper, a class of non-linear vector differential equations of third order with delay is considered. The stability, boundedness and ultimately boundedness of solutions are studied. The technique of proofs involves defining an appropriate Lyapunov functional. The obtained results include and improve the results in the literature.

1. Introduction

During the last years, many good results have been obtained on the qualitative behaviors in ordinary and functional differential equations of third order without and with delay. In particular, for some works on the stability and boundedness in scalar ordinary and functional differential equations of third order without and with delay, we referee the interested reader to the papers of Ademola et al. (2015), Ademola and Arawomo (2011), Afuwape and Castellanos (2010), Graef et al. (2015), Graef and Tunc (2015), Mehri and Shadman (1999), Meng (1993), Omeike (2014), Omeike and Afuwape (2010), Qian (2000), Remili and Oudjedi (2014), Tunc (2004, 2005a,b,c, 2007, 2009a,b, 2010a,b, 2013a,b, 2014, 2015), Tunc and Mohammed (2014), Tunc and Ateş (2006), Zhang and Yu (2013) and their references. However, to the best of our knowledge from the literature, by this time, little attention was given to the investigation into the stability/boundedness/ultimately boundedness in vector functional differential equations of third order with delay (see Tunc and Mohammed (2014)).

It should be noted any investigation into the stability and boundedness in vector functional differential equations of third order, using the Lyapunov functional method, first requires the definition or construction of a suitable Lyapunov functional, which gives meaningful results. In reality, this case can be an arduous task. The situation becomes more difficult when we replace an ordinary differential equation with a functional vector differential equation. However, once a viable Lyapunov functional has been defined or constructed, researchers may end up with working with it for a long time, deriving more information about stability. To arrive at the objective of this paper, we define a new suitable Lyapunov functional.

Recently, the authors in Tunc and Mohammed (2014) discussed the stability and boundedness in non-linear vector differential equation of third order with constant delay $\tau_1 > 0$:

$$X''' + \Psi(X')X'' + BX'(t-\tau_1) + cX(t-\tau_1) = P(t) \tag{1}$$

In this paper, we consider vector differential equation of third order of the form

$$X''' + H(X')X'' + G(X'(t-\tau)) + cX(t-\tau) = F(t, X, X', X'') \tag{2}$$

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where $\tau > 0$ is the fixed constant delay, c is a positive constant; $G : \Re^n \to \Re^n$ is a continuous differentiable function with $G(0) = 0$ and H is an $n \times n-$ continuous differentiable symmetric matrix function such that the Jacobian matrices $J_H(X')$ and $J_G(X')$ exist and are symmetric and continuous, that is,

$$J_H(X') = \left(\frac{\partial h_{ik}}{\partial x'_j}\right), \quad J_G(X') = \left(\frac{\partial g_i}{\partial x'_j}\right), \quad (i,j,k = 1,2,\ldots,n)$$

exist and are symmetric and continuous, where $(x'_1, x'_2, \ldots, x'_n)$, (h_{ik}) and (g_i) are components of X', H and G, respectively; $F : \Re^+ \times \Re^n \times \Re^n \times \Re^n \to \Re^n$ is a continuous function, $\Re^+ = [0, \infty)$, and the primes in Eq. (2) indicate differentiation with respect to t, $t \geqslant t_0 \geqslant 0$.

It should be stated that the continuity of the functions H, G and F is a sufficient condition for existence of the solution of Eq. (2). In addition, we assume that the functions H, G and F satisfy a Lipschitz condition with respect to their respective arguments, like X, X' and X''. In this case, the uniqueness of solutions of Eq. (2) is guaranteed.

It will be convenient here to consider not Eq. (2) itself, but rather the system

$$X'_1 = X_2, \quad X'_2 = X_3$$

$$X'_3 = -H(X_2)X_3 - G(X_2) + \int_{t-\tau}^{t} J_G(X_2(s))X_3(s)ds - cX_1 + c\int_{t-\tau}^{t} X_2(s)ds + F(t, X_1, X_2, X_3) \qquad (3)$$

derived from it by setting $X = X_1$, $X' = X_2$, $X'' = X_3$.

Along this paper, we assume that the existence and the uniqueness of the solutions of Eq. (2) hold.

The motivation of this paper comes from the results established in Datko (1994), De la Sen (1988a,b), De la Sen and Luo (2004), Omeike and Afuwape (2010), Qian (2000), Tunc and Mohammed (2014), Zhang and Yu (2013), the mentioned papers and their references. The main purpose of this paper is to get some new stability/boundedness/ultimately boundedness results in Eq. (1) using the Lyapunov-functional approach. By this paper, we will extend and improve the results of Omeike (2014), Tunc (2009b), Tunc and Mohammed (2014), Zhang and Yu (2013).

This is the novelty of this work. Besides, the results to be established here may be useful for researchers working on the qualitative behaviors of solutions.

One basic tool to be used here is LaSalle's invariance principle. Let us consider delay differential system

$$\dot{x} = f(x_t), \quad x_t = x(t+\theta), \quad -r \leqslant \theta \leqslant 0, \ t \geqslant 0$$

We take $C = C([-r, 0], \Re^n)$ to be the space of continuous function from $[-r, 0]$ into $\Re^n$ and ask that $f : C \to \Re^n$ be continuous. We say that $V : C \to \Re$ is a Lyapunov function on a set $G \subset C$ relative to f if V is continuous on $\bar{G}$, the closure of G, $\dot{V}$ is defined on G, and $\dot{V} \leqslant 0$ on G.

The following form of the LaSalle's invariance principle can be found in Tunc and Mohammed (2014).

Theorem A. *If V is a Lyapunov function on G and $x_t(\phi)$ is a bounded solution such that $x_t(\phi) \in G$ for $t \geqslant 0$, then $\omega(\phi) \neq 0$ is contained in the largest invariant subset of $E \equiv \{\psi \in \bar{G} : \dot{V}(\psi) = 0\}$, ω denotes the omega limit set of a solution.*

We need the following lemmas in the proofs of main results.

Lemma A. Hale (1965) *suppose $f(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \leqslant s < \infty$, $u(s) \to \infty$ as $u \to \infty$ with $u(0) = 0$. If for all $\varphi \in C$, $u(|\varphi(0)|) \leqslant V(\varphi)$, $V(\varphi) \geqslant 0$, $\dot{V}(\varphi) \leqslant 0$, then the zero solution of $\dot{x} = f(x_t)$ is stable.*

If we define $Z = \{\varphi \in C_H : \dot{V}(\varphi) = 0\}$, then the zero solution of $\dot{x} = f(x_t)$ is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

Lemma B. *Let A be a real symmetric $n \times n$-matrix. Then for any $X_1 \in \Re^n$*

$$\delta_a \|X_1\|^2 \leqslant \langle AX_1, X_1 \rangle \leqslant \Delta_a \|X_1\|^2$$

where δ_a and Δ_a are, respectively, the least and greatest eigenvalues of the matrix A.

2. Stability

Let $F(\cdot) \equiv 0$. The stability result of this paper is the following theorem.

Theorem 1. *In addition to the basic assumptions imposed on H G and c with $F(\cdot) \equiv 0$, we assume that there exist positive constants α, ε, a_0, a_1, b_0, b_1 and c such that the following conditions hold:*

$G(0) = 0$, J_G exists, $n \times n$-symmetric matrices J_G and H commute with each other,

$$a_0 b_0 - c > 0, \quad 1 - \alpha a_0 > 0, \quad b_0 \leqslant \lambda_i(J_G(X_2)) \leqslant b_1$$

and

$$a_0 + \varepsilon \leqslant \lambda_i(H(X_2)) \leqslant a_1 \text{ for all } X_2 \in \Re^n$$

If

$$\tau < \min\left\{\frac{\alpha a_0 b_0 c}{\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c}, \frac{k_5}{(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1}, \frac{k_6}{c + (2 + a_0 + \alpha a_0 b_0)b_1}\right\}$$

with

$$k_5 = 2(a_0 b_0 - c) - \alpha a_0 b_0 \left[a_0 + c^{-1}(b_1 - b_0)^2\right] > 0$$

and

$$k_6 = 2\varepsilon\left[1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2\right] > 0$$

then all solutions of Eq. (2) *are bounded and the zero solution of* Eq. (2) *is asymptotically stable.*

Proof. We define a functional $W(t) = W(X_1(t), X_2(t), X_3(t))$ given by

$$2W = a_0c\langle X_1, X_1\rangle + 2a_0\int_0^1 \langle \sigma H(\sigma X_2)X_2, X_2\rangle d\sigma$$
$$+ \alpha a_0 b_0^2\langle X_1, X_1\rangle + 2\int_0^1 \langle G(\sigma X_2), X_2\rangle d\sigma + \langle X_3, X_3\rangle$$
$$+ 2\alpha a_0^2 b_0\langle X_1, X_2\rangle + 2\alpha a_0 b_0\langle X_1, X_3\rangle + 2a_0\langle X_2, X_3\rangle$$
$$+ 2c\langle X_1, X_2\rangle - \alpha a_0 b_0\langle X_2, X_2\rangle + 2\lambda\int_{-\tau}^0\int_{t+s}^t \|X_2(\theta)\|^2 d\theta ds$$
$$+ 2\eta\int_{-\tau}^0\int_{t+s}^t \|X_3(\theta)\|^2 d\theta ds, \quad (4)$$

where

$$0 < \alpha < \min\left\{\frac{1}{a_0}, \frac{a_0}{b_0}, \frac{a_0b_0 - c}{a_0b_0\left[a_0 + c^{-1}(b_1 - b_0)^2\right]}, \frac{c}{a_0b_0(a_1 - a_0)}\right\} \quad (5)$$

$a_1 > a_0$, $b_1 \neq b_0$, and λ and η are positive constants which will be determined in the proof.

Since

$$G(0) = 0, \quad \frac{\partial}{\partial\sigma}G(\sigma X_2) = J_G(\sigma X_2)X_2$$

it follows that

$$\int_0^1 \langle G(\sigma X_2), X_2\rangle d\sigma = \int_0^1\int_0^1 \sigma_1\langle J_G(\sigma_1\sigma_2 X_2)X_2, X_2\rangle d\sigma_1 d\sigma_2$$

Then, from (4), we have clearly

$$2W = a_0b_0\left\|a_0^{-\frac{1}{2}}X_2 + a_0^{-\frac{1}{2}}b_0^{-1}cX_1\right\|^2 + \|X_3 + a_0X_2 + \alpha a_0b_0X_1\|^2$$
$$+ 2a_0\int_0^1 \langle \sigma H(\sigma X_2)X_2, X_2\rangle d\sigma - 2a_0^2\|X_2\|^2$$
$$+ 2\int_0^1\int_0^1 \sigma_1\langle J_G(\sigma_1\sigma_2X_2)X_2, X_2\rangle d\sigma_1 d\sigma_2 - b_0\|X_2\|^2$$
$$+ \alpha a_0b_0^2(1 - \alpha a_0)\|X_1\|^2 + c\left(a_0 - cb_0^{-1}\right)\|X_1\|^2$$
$$+ a_0(a_0 - \alpha b_0)\|X_2\|^2 + 2\lambda\int_{-\tau}^0\int_{t+s}^t \|X_2(\theta)\|^2 d\theta ds$$
$$+ 2\eta\int_{-\tau}^0\int_{t+s}^t \|X_3(\theta)\|^2 d\theta ds. \quad (6)$$

Under the hypotheses of Theorem 1, we have

$$W(0, 0, 0) = 0$$

$$2a_0\int_0^1 \langle \sigma H(\sigma X_2)X_2, X_2\rangle d\sigma - 2a_0^2\|X_2\|^2 \geqslant \varepsilon a_0\|X_2\|^2$$

$$2\int_0^1\int_0^1 \sigma_1\langle J_G(\sigma_1\sigma_2X_2)X_2, X_2\rangle d\sigma_1 d\sigma_2 - b_0\|X_2\|^2 \geqslant 0$$

$$\alpha a_0b_0^2(1 - \alpha a_0)\|X_1\|^2 = \mu_1\|X_1\|^2$$

$$\mu_1 = \alpha a_0b_0^2(1 - \alpha a_0) > 0$$

$$c\left(a_0 - cb_0^{-1}\right)\|X_1\|^2 = \mu_2\|X_1\|^2$$

$$\mu_2 = c\left(a_0 - cb_0^{-1}\right) > 0$$

$$a_0(a_0 - \alpha b_0)\|X_2\|^2 = \mu_3\|X_2\|^2$$

$$\mu_3 = a_0(a_0 - \alpha b_0) > 0$$

In summary, in view of (6), the above estimates imply that

$$W \geqslant \frac{1}{2}a_0b_0\left\|a_0^{-\frac{1}{2}}X_2 + a_0^{-\frac{1}{2}}b_0^{-1}cX_1\right\|^2 + \frac{1}{2}\|X_3 + a_0X_2 + \alpha a_0b_0X_1\|^2$$
$$+ \frac{1}{2}(\mu_1 + \mu_2)\|X_1\|^2 + \frac{1}{2}(a_0\varepsilon + \mu_3)\|X_2\|^2$$
$$+ 2\lambda\int_{-\tau}^0\int_{t+s}^t \|X_2(\theta)\|^2 d\theta ds + 2\eta\int_{-\tau}^0\int_{t+s}^t \|X_3(\theta)\|^2 d\theta ds.$$

It is clear from the first four terms that there exist sufficiently small positive constants k_i, $(i = 1, 2, 3)$, such that

$$W \geqslant k_1\|X_1\|^2 + k_2\|X_2\|^2 + k_3\|X_3\|^2$$

Let

$$k_4 = \min\{k_1, k_2, k_3\}$$

so that

$$W \geqslant k_4\left(\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2\right)$$

A straightforward calculation from (3) and (4) gives that

$$\dot{W}(t) = -\alpha a_0b_0c\|X_1\|^2 - a_0\langle X_2, G(X_2)\rangle + c\|X_2\|^2$$
$$+ \alpha a_0^2b_0\|X_2\|^2 - \alpha a_0b_0\langle X_1, H(X_2)X_3\rangle$$
$$+ \alpha a_0^2b_0\langle X_1, X_3\rangle - \langle H(X_2)X_3, X_3\rangle$$
$$+ a_0\|X_3\|^2 - \alpha a_0b_0\langle X_1, G(X_2)\rangle$$
$$+ \alpha a_0b_0^2\langle X_1, X_2\rangle + \left\langle X_3, \int_{t-\tau}^t J_G(X_2(s))X_3(s)ds\right\rangle$$
$$+ \left\langle X_3, c\int_{t-\tau}^t X_2(s)ds\right\rangle + \alpha a_0b_0\left\langle X_1, \int_{t-\tau}^t J_G(X_2(s)X_3(s)ds\right\rangle$$
$$+ \alpha a_0b_0c\left\langle X_1, \int_{t-\tau}^t X_2(s)ds\right\rangle + a_0\left\langle X_2, \int_{t-\tau}^t J_G(X_2(s))X_3(s)ds\right\rangle$$
$$+ a_0c\left\langle X_2, \int_{t-\tau}^t X_2(s)ds\right\rangle + \lambda\tau\|X_2\|^2 + \eta\tau\|X_3\|^2$$
$$- \lambda\int_{t-\tau}^t \|X_2(\theta)\|^2 d\theta - \eta\int_{t-\tau}^t \|X_3(\theta)\|^2 d\theta$$
$$= -\frac{1}{2}\alpha a_0b_0c\|X_1\|^2 - \langle (a_0G(X_2), X_2\rangle$$
$$+ \left\langle \left(cI + \alpha a_0^2b_0I\right)X_2, X_2\right\rangle - \langle (H(X_2) - a_0I)X_3, X_3\rangle$$
$$- \frac{1}{4}\alpha a_0b_0\left\|c^{\frac{1}{2}}X_1 + 2c^{-\frac{1}{2}}(H(X_2) - a_0I)X_3\right\|^2$$
$$+ \frac{1}{4}\alpha a_0b_0\left\|2c^{-\frac{1}{2}}(H(X_2) - a_0I)X_3\right\|^2$$
$$- \frac{1}{4}\alpha a_0b_0\left\|c^{\frac{1}{2}}X_1 + 2c^{-\frac{1}{2}}(G(X_2)X_2 - b_0X_2)\right\|^2$$
$$+ \frac{1}{4}\alpha a_0b_0\left\|2c^{-\frac{1}{2}}(G(X_2)X_2 - b_0X_2)\right\|^2$$
$$+ \left\langle X_3, \int_{t-\tau}^t J_G(X_2(s))X_3(s)ds\right\rangle + \left\langle X_3, c\int_{t-\tau}^t X_2(s)ds\right\rangle$$
$$+ \alpha a_0b_0\left\langle X_1, \int_{t-\tau}^t J_G(X_2(s))X_3(s)ds\right\rangle$$
$$+ \alpha a_0b_0c\left\langle X_1, \int_{t-\tau}^t X_2(s)ds\right\rangle + a_0\left\langle X_2, \int_{t-\tau}^t J_G(X_2(s)X_3(s)ds\right\rangle$$
$$+ a_0c\left\langle X_2, \int_{t-\tau}^t X_2(s)ds\right\rangle + \lambda\tau\|X_2\|^2 + \eta\tau\|X_3\|^2$$
$$- \lambda\int_{t-\tau}^t \|X_2(\theta)\|^2 d\theta - \eta\int_{t-\tau}^t \|X_3(\theta)\|^2 d\theta$$

The assumptions of Theorem 1 lead to

$$\langle a_0 G(X_2), X_2\rangle = \int_0^1 \langle a_0 J_G(\sigma X_2)X_2, X_2\rangle d\sigma \geqslant \int_0^1 \langle a_0 b_0 X_2, X_2\rangle d\sigma = a_0 b_0 \|X_2\|^2$$

$$a_0\langle X_2, G(X_2)\rangle - c\langle X_2, X_2\rangle - \alpha a_0^2 b_0\langle X_2, X_2\rangle \geqslant \left(a_0 b_0 - c - \alpha a_0^2 b_0\right)\|X_2\|^2$$

$$\begin{aligned}\left\langle X_3, \int_{t-\tau}^t J_G(X_2(s))X_3(s)ds\right\rangle &\leqslant \|X_3\|\int_{t-\tau}^t \|J_G(X_2(s))\|\|X_3(s)\|ds\\ &\leqslant b_1\|X_3\|\int_{t-\tau}^t \|X_3(s)\|ds\\ &\leqslant \frac{1}{2}b_1\int_{t-\tau}^t\left\{\|X_3(t)\|^2+\|X_3(s)\|^2\right\}ds\\ &= \frac{1}{2}b_1\tau\|X_3\|^2+\frac{1}{2}b_1\int_{t-\tau}^t\|X_3(s)\|^2ds\end{aligned}$$

$$\begin{aligned}\left\langle X_3, c\int_{t-\tau}^t X_2(s)ds\right\rangle &\leqslant c\|X_3\|\int_{t-\tau}^t\|X_2(s)\|ds\\ &\leqslant \frac{1}{2}c\tau\|X_3\|^2+\frac{1}{2}c\int_{t-\tau}^t\|X_2(s)\|^2ds\end{aligned}$$

$$\begin{aligned}&\alpha a_0 b_0\left\langle X_1, \int_{t-\tau}^t J_G(X_2(s))X_3(s)ds\right\rangle\\ &\leqslant \alpha a_0 b_0\|X_1\|\int_{t-\tau}^t\|J_G(X_2(s))\|\|X_3(s)\|ds\\ &\leqslant \frac{1}{2}\alpha a_0 b_0 b_1\int_{t-\tau}^t\left\{\|X_1(t)\|^2+\|X_3(s)\|^2\right\}ds\\ &= \frac{1}{2}\alpha a_0 b_0 b_1\tau\|X_1\|^2+\frac{1}{2}\alpha a_0 b_0 b_1\int_{t-\tau}^t\|X_3(s)\|^2ds\end{aligned}$$

$$\begin{aligned}\alpha a_0 b_0 c\left\langle X_1, \int_{t-\tau}^t X_2(s)ds\right\rangle &\leqslant \alpha a_0 b_0 c\|X_1\|\int_{t-\tau}^t\|X_2(s)\|ds\\ &\leqslant \frac{1}{2}\alpha a_0 b_0 c\tau\|X_1\|^2+\frac{1}{2}\alpha a_0 b_0 c\\ &\quad\times\int_{t-\tau}^t\|X_2(s)\|^2ds\end{aligned}$$

$$\begin{aligned}a_0\left\langle X_2, \int_{t-\tau}^t J_G(X_2(s))X_3(s)ds\right\rangle &\leqslant a_0 b_1\|X_2\|\int_{t-\tau}^t\|J_G(X_2(s))\|\|X_3(s)\|ds\\ &\leqslant \frac{1}{2}a_0 b_1\tau\|X_2\|^2+\frac{1}{2}a_0 b_1\int_{t-\tau}^t\|X_3(s)\|^2ds\end{aligned}$$

$$\begin{aligned}a_0 c\left\langle X_2, \int_{t-\tau}^t X_2(s)ds\right\rangle &\leqslant a_0 c\|X_2\|\int_{t-\tau}^t\|X_2(s)\|ds\\ &\leqslant \frac{1}{2}a_0 c\int_{t-\tau}^t\left\{\|X_2(t)\|^2+\|X_2(s)\|^2\right\}ds\\ &= \frac{1}{2}a_0 c\tau\|X_2\|^2+\frac{1}{2}a_0 c\int_{t-\tau}^t\|X_2(s)\|^2ds\end{aligned}$$

On combining the above obtained inequalities into $\dot{W}(t)$, we have that

$$\begin{aligned}\dot{W}(t) &\leqslant -\frac{1}{2}\alpha a_0 b_0 c\|X_1\|^2-\left(a_0 b_0-c-\alpha a_0^2 b_0\right)\|X_2\|^2\\ &-\langle (H(X_2)-a_0 I)X_3, X_3\rangle\\ &-\frac{1}{4}\alpha a_0 b_0\left\|c^{\frac{1}{2}}X_1+2c^{-\frac{1}{2}}(H(X_2)-a_0 I)X_3\right\|^2\\ &+\frac{1}{4}\alpha a_0 b_0\left\|2c^{-\frac{1}{2}}(H(X_2)-a_0 I)X_3\right\|^2\\ &-\frac{1}{4}\alpha a_0 b_0\left\|c^{\frac{1}{2}}X_1+2c^{-\frac{1}{2}}(B-b_0 I)X_2\right\|^2\\ &+\frac{1}{4}\alpha a_0 b_0\left\|2c^{-\frac{1}{2}}(B-b_0 I)X_2\right\|^2\\ &+\frac{1}{2}\alpha a_0 b_0 b_1\tau\|X_1\|^2+\frac{1}{2}\alpha a_0 b_0 c\tau\|X_1\|^2\\ &+\frac{1}{2}a_0 b_1\tau\|X_2\|^2+\frac{1}{2}a_0 c\tau\|X_2\|^2\\ &+\frac{1}{2}b_1\tau\|X_3\|^2+\frac{1}{2}c\tau\|X_3\|^2+\lambda\tau\|X_2\|^2+\eta\tau\|X_3\|^2\\ &-\left\{\lambda-\frac{1}{2}(a_0+\alpha a_0 b_0+1)c\right\}\int_{t-\tau}^t\|X_2(s)\|^2ds\\ &-\left\{\eta_1-(1+a_0+\frac{1}{2}\alpha a_0 b_0)b_1\right\}\int_{t-\tau}^t\|X_3(s)\|^2ds\end{aligned}$$

Let

$$\lambda=\frac{1}{2}(a_0+\alpha a_0 b_0+1)c \text{ and } \eta=\left(1+a_0+\frac{1}{2}\alpha a_0 b_0\right)b_1$$

Hence

$$\begin{aligned}\dot{W}(t) &\leqslant -\frac{1}{2}\alpha a_0 b_0 c\|X_1\|^2-\left(a_0 b_0-c-\alpha a_0^2 b_0\right)\|X_2\|^2\\ &-\langle (H(X_2)-a_0 I)X_3, X_3\rangle\\ &+\frac{1}{4}\alpha a_0 b_0\left\|2c^{-\frac{1}{2}}(H(X_2)-a_0 I)X_3\right\|^2\\ &+\frac{1}{4}\alpha a_0 b_0\left\|2c^{-\frac{1}{2}}(B-b_0 I)X_2\right\|^2\\ &+\frac{1}{2}(\alpha a_0 b_0 b_1+\alpha a_0 b_0 c)\tau\|X_1\|^2+\frac{1}{2}(a_0 b_1+a_0 c)\tau\|X_2\|^2\\ &+\frac{1}{2}(a_0+\alpha a_0 b_0+1)c\tau_1\|X_2\|^2+\frac{1}{2}(b_1+c)\tau\|X_3\|^2\\ &+\frac{1}{2}(1+a_0+\alpha a_0 b_0)b_1\tau\|X_3\|^2\end{aligned}$$

Since

$$\frac{1}{4}\alpha a_0 b_0\left\|2c^{-\frac{1}{2}}(B-b_0 I)X_2\right\|^2=\alpha a_0 b_0\langle c^{-1}(B-b_0 I)X_2, (B-b_0 I)X_2\rangle$$

and

$$\frac{1}{4}\alpha a_0 b_0\left\|2c^{-\frac{1}{2}}(H(X_2)-a_0 I)X_3\right\|^2=\alpha a_0 b_0\langle c^{-1}(H(X_2)-a_0 I)X_3, (H(X_2)-a_0 I)X_3\rangle$$

it is clear that

$$\dot{W}(t) \leqslant -\frac{1}{2}\alpha a_0 b_0 c\|X_1\|^2 - \left(a_0 b_0 - c - \alpha a_0^2 b_0\right)\|X_2\|^2$$
$$+ \alpha a_0 b_0 \left\langle c^{-1}(B - b_0 I)X_2, (B - b_0 I)X_2\right\rangle$$
$$- \langle (H(X_2) - a_0 I)X_3, X_3\rangle$$
$$+ \alpha a_0 b_0 \left\langle c^{-1}(H(X_2) - a_0 I)X_3, (H(X_2) - a_0 I)X_3\right\rangle$$
$$+ \frac{1}{2}(\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c)\tau\|X_1\|^2 + \frac{1}{2}(a_0 b_1 + a_0 c)\tau\|X_2\|^2$$
$$+ \frac{1}{2}(a_0 + \alpha a_0 b_0 + 1)c\tau\|X_2\|^2 + \frac{1}{2}(b_1 + c)\tau\|X_3\|^2$$
$$+ \frac{1}{2}(1 + a_0 + \alpha a_0 b_0)b_1\tau\|X_3\|^2$$

By Lemma B and the assumptions of Theorem 1, we get

$$\dot{W}(t) \leqslant -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c)\tau\}\|X_1\|^2$$
$$- \left\langle \left\{(a_0 B - cI) - \alpha a_0 b_0\left[a_0 I + c^{-1}(B - b_0 I)^2\right]\right\}X_2, X_2\right\rangle$$
$$+ \frac{1}{2}(a_0 b_1 + a_0 c)\tau_1\|X_2\|^2$$
$$+ \frac{1}{2}(a_0 + \alpha a_0 b_0 + 1)c\tau_1\|X_2\|^2$$
$$- \left\langle \{(H(X_2) - a_0 I)[I - \alpha a_0 b_0 c^{-1}(H(X_2) - a_0 I)]\}X_3, X_3\right\rangle$$
$$+ \frac{1}{2}(b_1 + c)\tau\|X_3\|^2$$
$$+ \frac{1}{2}(1 + a_0 + \alpha a_0 b_0)b_1\tau\|X_3\|^2$$
$$\leqslant -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c)\tau\}\|X_1\|^2$$
$$- \left\{(a_0 b_0 - c) - \alpha a_0 b_0[a_0 + c^{-1}(b_1 - b_0)^2]\right\}\|X_2\|^2$$
$$+ \frac{1}{2}\{(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1\}\tau\|X_2\|^2$$
$$- \varepsilon\left[1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2\right]\}\|X_3\|^2$$
$$+ \frac{1}{2}(2b_1 + c + a_0 b_1 + \alpha a_0 b_0 b_1)\tau\|X_3\|^2.$$

Let

$$k_5 = 2(a_0 b_0 - c) - \alpha a_0 b_0\left[a_0 + c^{-1}(b_1 - b_0)^2\right] > 0$$

and

$$k_6 = 2\varepsilon\left[1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2\right] > 0$$

so that

$$\dot{W}(t) \leqslant -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c)\tau\}\|X_1\|^2$$
$$-\frac{1}{2}\{k_5 - [(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1]\tau\}\|X_2\|^2$$
$$-\frac{1}{2}\{k_6 - (2b_1 + c + a_0 b_1 + \alpha a_0 b_0 b_1)\tau\}\|X_3\|^2$$

If

$$\tau < \min\left\{\frac{\alpha a_0 b_0 c}{\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c}, \frac{k_5}{(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1}, \frac{k_6}{c + (2 + a_0 + \alpha a_0 b_0)b_1}\right\}$$

then, for some positive constants k_7, k_8 and k_9, it follows that

$$\dot{W}(t) \leqslant -k_7\|X_1\|^2 - k_8\|X_2\|^2 - k_9\|X_3\|^2 \leqslant 0$$

In addition, we can easily see that

$$W(X_1, X_2, X_3) \to \infty \text{ as } \|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \to \infty$$

Consider the set defined by

$$E \equiv \left\{(X_1, X_2, X_3) : \dot{W}(X_1, X_2, X_3) = 0\right\}$$

When we apply LaSalle's invariance principle, we observe that $(X_1, X_2, X_3) \in E$ implies that $X_1 = X_2 = X_3 = 0$. Clearly, this fact leads that the largest invariant set contained in E is $(0, 0, 0) \in E$. By Lemma B, we conclude that the zero solution of system (3) is asymptotically stable. Hence, the zero solution of Eq. (2) is asymptotically stable. This completes the proof of Theorem 1.

3. Boundedness

Let $F(\cdot) \neq 0$. The boundedness result of this paper is the following theorem.

Theorem 2. *We assume that all the assumptions of* Theorem 1 *hold, except $F(\cdot) \equiv 0$. Further, we suppose that there exists a non-negative and continuous function $\theta = \theta(t)$ such that*

$$\|F(t, X_1, X_2, X_3)\| \leqslant \theta(t) \text{ for all } t \geqslant 0, \ \max\theta(t) < \infty \text{ and } \theta \in L^1(0, \infty)$$

where $L^1(0,\infty)$ denotes the space of Lebesgue integrable functions.

If

$$\tau < \min\left\{\frac{\alpha a_0 b_0 c}{\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c}, \frac{k_5}{(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1}, \frac{k_6}{c + (2 + a_0 + \alpha a_0 b_0)b_1}\right\}$$

with

$$k_5 = 2(a_0 b_0 - c) - \alpha a_0 b_0\left[a_0 + c^{-1}(b_1 - b_0)^2\right] > 0$$

and

$$k_6 = 2\varepsilon\left[1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2\right] > 0$$

then there exists a constant $D > 0$ such that any solution $(X_1(t), X_2(t), X_3(t))$ of system (3) *determined by*

$$X_1(0) = X_{10}, \quad X_2(0) = X_{20}, \quad X_3(0) = X_{30}$$

Satisfies

$$\|X_1(t)\| \leqslant D, \quad \|X_2(t)\| \leqslant D, \quad \|X_3(t)\| \leqslant D$$

for all $t \in \Re^+$.

Proof. Let $F(\cdot) = F(t, X_1, X_2, X_3)$. In the case of $F(\cdot) \neq 0$, under the assumptions of Theorem 2, we can easily arrive at

$$\dot{W}(t) \leqslant -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c)\tau\}\|X_1\|^2$$
$$-\frac{1}{2}\{k_5 - [(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1]\tau\}\|X_2\|^2$$
$$-\frac{1}{2}\{k_6 - (2b_1 + c + a_0 b_1 + \alpha a_0 b_0 b_1)\tau\}\|X_3\|^2$$
$$+ \langle X_3, F(\cdot)\rangle + \alpha a_0 b_0\langle X_1, F(\cdot)\rangle + a_0\langle X_2, F(\cdot)\rangle$$
$$\leqslant (\alpha a_0 b_0\|X_1\| + a_0\|X_2\| + \|X_3\|)\|F(\cdot)\|$$
$$\leqslant \sigma(\|X_1\| + \|X_2\| + \|X_3\|)\|F(\cdot)\|$$
$$\leqslant \sigma(3 + \|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2)\theta(t)$$

where

$\sigma = \max\{\alpha a_0 b_0, a_0, 1\}$

Besides, in view of the discussion made, it is clear that

$$\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \leqslant k_4^{-1} W$$

so that

$$\dot{W}(t) \leqslant 3\sigma\theta(t) + k_4^{-1} W(t)\theta(t)$$

Integrating both sides of the last estimate from 0 to t ($t \geqslant 0$), we have

$$W(t) \leqslant W(0) + 3\sigma \int_0^t \theta(s)ds + k_4^{-1} \int_0^t W(s)\theta(s)ds$$

Let

$$M = W(0) + 3\sigma \int_0^\infty \theta(s)ds$$

Then

$$W(t) \leqslant M + k_4^{-1} \int_0^\infty W(s)\theta(s)ds$$

By noting the Gronwall–Bellman inequality, we can get

$$W(t) \leqslant M \exp\left(k_4^{-1} \int_0^\infty \theta(s)ds\right)$$

By the estimate $\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \leqslant k_4^{-1} W$ and the assumption $\theta \in L^1(0, \infty)$, we can conclude that all solutions of system (2) are bounded. This completes the proof of Theorem 2.

4. Ultimately boundedness

For the case $F(\cdot) \neq 0$, the ultimately boundedness result of this paper is the following theorem.

Theorem 3. *We assume that all assumptions of Theorem 1 hold, except $F(\cdot) \equiv 0$. In addition, we assume that there exists a positive constant δ_0 such that the condition*

$$\|F(\cdot)\| \leqslant \delta_0, \quad (t \geqslant 0)$$

holds.

If

$$\tau < \min\left\{\frac{\alpha a_0 b_0 c}{\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c}, \frac{k_5}{(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1}, \frac{k_6}{c + (2 + a_0 + \alpha a_0 b_0) b_1}\right\},$$

With

$$k_5 = 2(a_0 b_0 - c) - \alpha a_0 b_0 \left[a_0 + c^{-1}(b_1 - b_0)^2\right] > 0$$

and

$$k_6 = 2\varepsilon\left[1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2\right] > 0$$

then there exists a constant $d > 0$ such that any solution $(X_1(t), X_2(t), X_3(t))$ of system (3) determined by

$$X_1(0) = X_{10}, \quad X_2(0) = X_{20}, \quad X_3(0) = X_{30}$$

ultimately satisfies

$$\|X_1(t)\|^2 + \|X_2(t)\|^2 + \|X_3(t)\|^2 \leqslant k$$

for all $t \in \Re^+$.

Proof. For the case $F(\cdot) \neq 0$, in the light of the assumptions of Theorem 3, we can conclude that

$$\begin{aligned}\dot{W}(t) &\leqslant -\rho_1\|X_1\|^2 - \rho_2\|X_2\|^2 - \rho_3\|X_3\|^2 \\ &\quad + (\alpha a_0 b_0\|X_1\| + a_0\|X_2\| + \|X_3\|)\|F(\cdot)\| \\ &\leqslant -\rho_1\|X_1\|^2 - \rho_2\|X_2\|^2 - \rho_3\|X_3\|^2 \\ &\quad + (\alpha a_0 b_0 \delta_0\|X_1\| + a_0\delta_0\|X_2\| + \delta_0\|X_3\|)\end{aligned}$$

The rest of the proof can be easily done by following a similar procedure as shown in Meng (1993), Tunc and Mohammed (2014). Hence, we omit the details of the proof.

5. Conclusion

A kind of nonlinear vector functional differential equations of third order with a constant delay has been considered. Some qualitative behaviors of solutions, stability/boundedness/ultimately boundedness of solutions, have been discussed. The technique of proofs involves defining an appropriate Lyapunov functional. Our results include and improve some recent results in the literature.

Competing interests

The author declares that he has no competing interests.

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