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ORIGINAL ARTICLE

Solution of mixed integral equation in position and time using spectral relationships

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Abstract In this article, the existence of a unique solution of Fredholm–Volterra integral equation of the second kind is guaranteed. The Fredholm integral term is assumed in position with bad kernel, while the Volterra integral term is considered in time with continuous kernel. Under certain conditions and new discussions, the bad kernel will tend to a logarithmic kernel. Then, using Chebyshev polynomial, a main theorem of spectral relationships of Fredholm integral equation of the first kind with logarithmic kernel multiplying by a smooth kernel is stated and used to obtain numerically the Fredholm–Volterra integral equation of the second kind. Finally, numerical results are obtained and the error, in each case, is computed.

1. Introduction

Integral equations of various types and kinds are playing an important role in branches of mathematical physics (Bai, 2013), mathematical engineering (Assari et al., 2013) and contact problems in the theory of elasticity (Heydari et al., 2013; Li and HuaZou, 2013; Aleksandrov and Covalenko, 1986). Therefore, many different methods are established and used to solve the linear and nonlinear integral equation analytically and numerically (Abdou, 2002; Diogo and Lima, 2008; Anastassiou George and Ali, 2009; Bazm and Babolian, 2012;Biazar et al., 2003; Yüzbasi, 2014; Yüzbasi et al., 2011; Toutounian and Tohidi, 2013).

Here, Consider the F–VIE of second kind

$$\mu\Phi(x,t) - \lambda\int_0^t V(|t-\tau|)\Phi(x,\tau)d\tau - \lambda\int_{-1}^{1} p(x,y)k\left(\left|\frac{y-x}{c}\right|\right)\Phi(y,t)dy = \pi\theta[\gamma(t) - f_*(x)] = f(x,t), \quad (1.1)$$

$$k(z) = \int_0^\infty \frac{L(u)\cos(uz)}{u}du, \quad L(u) = \frac{u+q}{1+u}, \quad z = \frac{y-x}{c}, \quad q \geqslant 1,\ c \in (0,\infty) \quad (1.2)$$

under the pressure condition

$$\int_{-1}^{1} \phi(t,y)dy = P(t), \quad t \in [0,T],\ T < 1. \quad (1.3)$$

where $\theta = \frac{G}{2(1-v)}$ in which G is a modulus of elasticity and v is Poisson's coefficient and λ has many physical meanings. The constant μ defined the kind of integral equation. The given

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Table 1 Case (i-a, b); represents the solution $\Phi_N(x,t)$ and its error for different times in the simple case $m=0$.

t	x	Φ	Error
0.0004	−0.99	0.0000016329	$3.078945881\times10^{-10}$
	−0.495	0.0000003591	$4.857498762\times10^{-11}$
	0.495	0.0000002638	$2.189115679\times10^{-11}$
	0.99	0.0000006029	$5.164023745\times10^{-11}$
0.03	−0.99	0.009023297514	1.700838626×10^{-6}
	−0.495	0.001984376047	2.687898694×10^{-7}
	0.495	0.001457778739	1.199968349×10^{-7}
	0.99	0.003331872637	5.164023745×10^{-7}
0.8	−0.99	4.060805158	2.014803307×10^{-3}
	−0.495	0.8979563585	1.012343762×10^{-3}
	0.495	0.6898680429	1.23451387×10^{-3}
	0.99	1.503538143	2.24356819×10^{-3}

Table 2 Case (ii-c, d); represents the solution $\Phi_N(x,t)$ and its error for different times in $|n-m|\geqslant 1, m\neq 0, m\neq n$.

t	x	Φ	Error
0.0004	−0.99	0.0000001212	$4.537270489\times10^{-11}$
	−0.495	0.00000002348	$7.158224453\times10^{-12}$
	0.495	0.00000001831	$3.225977507\times10^{-12}$
	0.99	0.0000000455	$1.380206111\times10^{-12}$
0.03	−0.99	0.0006696992769	2.507393716×10^{-7}
	−0.495	0.0001297634539	3.955503532×10^{-8}
	0.495	0.0001011735411	1.783327980×10^{-8}
	0.99	0.0002514196828	7.627930927×10^{-8}
0.8	−0.99	0.4759821898	2.198124642×10^{-4}
	−0.495	0.09221639101	2.652620889×10^{-6}
	0.495	0.07185440241	8.092946383×10^{-5}
	0.99	0.1785797793	1.35594521×10^{-4}

Table 3 Case (ii-c, e); represents the solution $\Phi_N(x,t)$ and its error for different times in $(n+m)\geqslant 2, m\neq 0$.

t	x	Φ	Error
0.0004	−0.99	0.0000002684	$9.725059768\times10^{-11}$
	−0.495	0.00000005247	$1.534273982\times10^{-11}$
	0.495	0.00000004113	$6.914470741\times10^{-12}$
	0.99	0.00000010064	$2.958295512\times10^{-11}$
0.03	−0.99	0.0006625698160	2.507508228×10^{-7}
	−0.495	0.0001303995479	3.955688645×10^{-8}
	0.495	0.0001015636111	1.783400322×10^{-8}
	0.99	0.0002517222761	7.628269714×10^{-8}
0.8	−0.99	0.4709180393	2.191556523×10^{-4}
	−0.495	0.09266452116	3.060634234×10^{-6}
	0.495	0.07213350091	7.983953360×10^{-5}
	0.99	0.1788083769	1.342967242×10^{-4}

function of time $V(|t-\tau|)$ represents the kernel of VI term and belongs to the class $C([0,T],[0,T])$, where $t,\tau\in[0,T], T<1$. The kernel of position, $p(x,y)k\left(\left|\frac{y-x}{c}\right|\right)$, of FI term behaved badly for $k\left(\left|\frac{y-x}{c}\right|\right)$, given by (1.2), and smooth for $p(x,y)$. The given functions $f_*(x)$ belongs to the space $L_2[-1,1]$, while $\gamma(t), P(t)$ belong to the space $C[0,T]$. The integral equation (1.1) with bad kernel of position (1.2), under the pressure condition (1.3), is investigated from the mixed contact problem of a rigid elastic surface (G,v), G is the displacement magnitude and v is Poisson's coefficient, having an elastic material occupying the domain $[-1,1]$ with respect to position through the time $t, t\in[0,T], T<1$. The given function $f(x,t)$ is the sum of two functions, the first function $\gamma(t)$ represents the displacement of the surface under the action of pressure $P(t)$, and the second function $f_*(x)$ describes the basic formula of the surface. The unknown function $\Phi(x,t)$ represents the normal stresses between the layers of the surface, which is supplied by a position force $p(x,y)$.

This paper is divided into 7 sections. In section 2, the existence and uniqueness of Eq. (1.1) are discussed. In section 3 F–VIE of the second kind is considered in position and time in the space $L_2[-1,1]\times C[0,T], T<1$. The FI term belongs to the space $L_2[-1,1]$ and has a bad behavior kernel. While the VI term belongs to the space $C[0,T], T<1$, with a continuous kernel. In section 4, a main theorem of spectral relationships for the FIE of the first kind is considered. In section 5, we use the main theorem to obtain a linear system of VIE of the second kind. In section 6, numerical results and estimated errors are computed. In section 7, general conclusions are deduced.

2. Existence and uniqueness

In order to guarantee the existence of a unique solution of Eq. (1.1), under the pressure condition (1.3), we assume the following:

(i) The bad behaved kernel $k\left(\left|\frac{y-x}{c}\right|\right)$ satisfies $\left\{\int_{-1}^{1}\int_{-1}^{1}k^2\left(\left|\frac{y-x}{c}\right|\right)dxdy\right\}^{\frac{1}{2}}=A$, where A is a small constant, while the smooth kernel $|p(x,y)|<N_1$.

(ii) The positive kernel of time is continuous and satisfies
$V(|t-\tau|)<N_2, \quad \forall t,\tau\in[0,T]$.

(iii) The given function $f(x,t)$ is continuous with its partial derivatives in the space $L_2[-1,1]\times C[0,T]$ and its norm is given as $\|f\|_{L_2\times C}=\max_{0\leqslant t\leqslant T}\left\{\int_{-1}^{1}f^2(x,\tau)dx\right\}^{\frac{1}{2}}$.

(iv) The unknown function $\phi(x,t)$ satisfies Lipschitz condition with respect to position and Hölder condition with respect to time and behaves in $L_2[-1,1]\times C[0,T]$ as the given function $f(x,t)$.

Theorem 2.1. *The solution of Eq. (1.1) exists and is unique under the condition*

$$|\lambda|\leqslant\frac{|\mu|}{N_2T+N_1C}. \tag{2.4}$$

The existence of solutions will be proved using Picard method. For this, Eq. (1.1) becomes

$$\mu\Phi_n(x,t) = \lambda\int_0^t V(|t-\tau|)\Phi_{n-1}(x,\tau)d\tau + \lambda\int_{-1}^{1} p(x,y)k\left(\left|\frac{y-x}{c}\right|\right)\Phi_{n-1}(y,t)dy + f(x,t). \quad (2.5)$$

Introduce

$$\Psi_n(x,t) = \Phi_n(x,t) - \Phi_{n-1}(x,t), \quad \mu\Psi_0(x,t) = f(x,t), \quad (2.6)$$

where $\Psi_0 = \Phi_0$. Then, we deduce

$$\Phi_n(x,t) = \sum_{j=0}^{n}\Psi_j(x,t). \quad (2.7)$$

Hence, after using Cauchy–Schwarz inequality and applying the previous conditions, one obtains

$$\|\Psi_n(x,t)\| \leqslant \alpha\|\Psi_{n-1}(x,t)\|, \quad \alpha = \left|\frac{\lambda}{\mu}\right|(N_2T + N_1A). \quad (2.8)$$

By induction, we get

$$\|\Psi_n(x,t)\| = \alpha^n\frac{H}{\mu}. \quad (2.9)$$

Therefore, using the condition of Eq. (2.4), we deduce that $\{\Psi_n(x,t)\}$ is uniformly convergent, then the formula (2.7) is also uniformly convergent and leads to write

$$\Phi_n(x,t) = \sum_{j=0}^{\infty}\Psi_j(x,t). \quad (2.10)$$

Therefore, $\Phi(x,t)$ exists and represents a continuous solution.

To prove the uniqueness, assume $\bar{\Phi}(x,t)$ is another solution. Hence, we get

$$\|\bar{\Phi}(x,t)-\Phi(x,t)\| = \left|\frac{\lambda}{\mu}\right|\left|\int_0^t V(|t-\tau|)(\bar{\Phi}(x,\tau)-\Phi(x,\tau))d\tau + \lambda\int_{-1}^{1}p(x,y)k\left(\left|\frac{y-x}{c}\right|\right)(\bar{\Phi}(y,t)-\Phi(y,t))dy\right|. \quad (2.11)$$

Using Cauchy–Schwarz inequality, we have

$$\|\bar{\Phi}(x,t)-\Phi(x,t)\| \leqslant \alpha\|\bar{\Phi}(x,t)-\Phi(x,t)\|. \quad (2.12)$$

From (2.4), we see that $\bar{\Phi} = \Phi$.

3. The kernel of position

The function $L(u)$ of Eq. (1.2) is continuous and positive, for $u \in (0,\infty)$ and satisfies the following asymptotic equalities:

$$L(u) = q - (q-1)u + O(u^2), \quad u \to 0, \quad (3.13)$$

$$L(u) = 1 + \frac{q-1}{u} + O(u^{-2}), \quad u \to \infty, \ q \geqslant 1. \quad (3.14)$$

All the previous works of Popov (1982) and Aleksandrov and Covalenko (1986), in solving the problems of continuum mechanics, discussed the solution of FIE of the first and second kind when the kernel takes the form of Eq. (1.2) under the condition (3.14), i.e. $u \to \infty$ or $q = 1$.

In this work, we discuss the solution of F–VIE when the kernel of Fredholm in the form of Eq. (1.2) under the condition $u \to 0$ of Eq. (3.13). For this aim, consider the first and the second approximation of $L(u)$ in (3.13). After using the following famous relations (Gradshteyn and Ryzhik, 1994)

$$\int_0^{\infty}\frac{\cos(uz)}{u}du = -\ln|x-y| + d, \quad \left(d = \ln\frac{4\lambda}{\pi}, \ c \to \infty\right), \quad (3.15)$$

$$\frac{1}{\pi}\int_0^{\infty}\cos(vx)dv = \delta(x), \quad \delta(x) \text{ is the Dirac function}, \quad (3.16)$$

$$\int_a^b \psi(y)\delta(y-x)dy = \begin{cases}\frac{1}{2}[u(x-a)+u(x+b)], & a \leqslant x \leqslant b \\ 0, & otherwise.\end{cases} \quad (3.17)$$

where $\psi(y)$ is any function, the IE (2.5) will be reduced to the following

$$\bar{\mu}\Phi(x,t) - \lambda\int_0^t H(|t-\tau|)\Phi(x,\tau)d\tau + \bar{\lambda}\int_{-1}^{1}p(x,y)[\ln|x-y| - d]\Phi(y,t)dy + \int_{-1}^{1}p(x,y)\delta\left(\frac{y-x}{c}\right)\Phi(y,t)dy = \frac{1}{q-1}f(x,t). \quad (3.18)$$

Here,

$$\bar{\mu} = \frac{\mu}{q-1}, \quad H(|t-\tau|) = \frac{V(|t-\tau|)}{q-1}, \quad (q>1), \quad (3.19)$$

and

$$\bar{\lambda} = \frac{q\lambda}{q-1}. \quad (3.20)$$

From the values of $\bar{\lambda}$, we obtain the physical meaning between $\bar{\lambda}$ and q in which the logarithmic kernel satisfies

$$\left\{\int_{-1}^{1}\int_{-1}^{1}\ln^2|x-y|dxdy\right\}^{\frac{1}{2}} < \frac{1}{\bar{\lambda}}, \quad \bar{\lambda} = \lambda\left[1+\frac{1}{q}+\frac{1}{q^2}+\cdots\right]. \quad (3.21)$$

Also, from (3.19), we can deduce that for large values of q, the total resistance force $H(|t-\tau|)$ decreases. So, the external resistance force for $q \to \infty$, is not available and the total resistance, in this case, is the resistance of material only.

As an important special case, put $t = 0$ and $p(x,y) = 1$, in Eq. (3.18), then differentiating the result with respect to x and taking the substitution $y = 2u-1, x = 2v-1$, to get

$$\frac{d\Theta(u)}{du} - \lambda^*\int_0^1\frac{\Theta(v)dv}{u-v} = h(u), \quad (3.22)$$

$$\Theta(u) = \Psi(x) = \Phi(x,0), \quad h(u) = g(x) = \frac{f'(x,0)}{\mu(q-1)}, \quad \lambda^* = \frac{\bar{\lambda}}{\mu},$$

under the condition

$$\Theta(0) = \Theta(1) = 0,$$

Eq. (3.22) has appeared in both combined infrared gaseous radiation and molecular conduction.

4. Spectral relationships of FIE of the first kind

Theorem 4.2 (Main theorem). *The spectral relationships for the integral equation*

$$\int_{-1}^{1} p(x,y)[\ln|x-y|-d]\Phi(x,t)dy = f(x,t), \tag{4.23}$$

after representing the given functions $p(x,y), f(x,t)$ and the unknown function $\Phi(x,t)$ in terms of Chebyshev polynomials of the first kind of order n, $T_n(x)$, are given as

$$\int_{-1}^{1} \frac{p(x,y)[\ln|x-y|-d]T_n(y)}{\sqrt{(1-y^2)}}dy = \begin{cases} \pi(\ln 2-d), & n=m=0 \\ \frac{\pi}{n}T_n(x), & m=0,\ n\geqslant 1 \\ \pi\sum_{m=1}^{M}\frac{1-2m^2}{m(1-4m^2)}, & m\neq 0,\ n=0 \\ \frac{\pi}{4(|n-m|)}T_{|n-m|}(x), & |n-m|\geqslant 1,\ n\neq m\neq 0 \\ \frac{\pi}{4(n+m)}T_{n+m}(x), & n+m\geqslant 1,\ n\neq 0, m\neq 0, \end{cases} \tag{4.24}$$

where $p(x,y)$ is given by

$$p(x,y) = \sum_{m=0}^{M} T_m(x)T_m(y). \tag{4.25}$$

The proof of this theorem depends completely on the work of Abdou and Basseem (2010).

5. Method of solution

For solving Eq. (3.18), we assume that the given function $f(x,t)$, in the light of weight function of $T_n(x)$, is given by

$$f(x,t) = \frac{1}{\sqrt{1-x^2}}\sum_{n=0}^{\infty} f_n(t)T_n(x), \tag{5.26}$$

where $f_n(t)$ satisfies

$$f_n(t) = \begin{cases} \frac{2}{\pi}\int_{-1}^{1} f(x,t)T_n(x)dx, & n\neq 0 \\ \frac{1}{\pi}\int_{-1}^{1} f(x,t)dx, & n=0. \end{cases} \tag{5.27}$$

and the unknown function $\Phi(x,t)$ takes the form

$$\Phi(x,t) = \sum_{n=0}^{\infty}\frac{a_n(t)T_n(x)}{\sqrt{1-x^2}}. \tag{5.28}$$

To compute the error, the formula (5.28) can be truncated to

$$\Phi_N(x,t) = \sum_{n=0}^{N}\frac{a_n(t)T_n(x)}{\sqrt{1-x^2}}. \tag{5.29}$$

In this case, see (Abdou et al., 2009), the orthogonal polynomials method is convergent of order r if and only if for N sufficiently large number, there exists $D(t)>0, \forall t\in[0,T]$, independent of N such that

$$\|\Phi(x,t)-\Phi_N(x,t)\| \leqslant D(t)N^{-r}. \tag{5.30}$$

Hence, the truncated error is

$$E_N = \left\|\max_{0\leqslant t\leqslant T}\sum_{n=N+1}^{\infty} a_n(t)\right\| \leqslant D(t)N^{-r}. \tag{5.31}$$

With the aid of Eqs. (5.26), (4.25), (5.28), (3.17) and (3.16) the formula (3.18) reduced to

$$\bar{\mu}\sum_{n=0}^{N} a_n(t)\frac{T_n(x)}{\sqrt{1-x^2}} - \lambda\sum_{n=0}^{N}\int_0^t a_n(\tau)H(|t-\tau|)\frac{T_n(x)}{\sqrt{1-x^2}}d\tau$$
$$+\bar{\lambda}\sum_{n=0}^{N}\sum_{m=0}^{M}T_m(x)\int_{-1}^{1} a_n(t)T_m(y)(\ln|x-y|-d)$$
$$\times\frac{T_n(y)}{\sqrt{1-y^2}}dy + a_n(t)\sum_{m=0}^{M}T_m(x) = \frac{1}{q-1}\sum_{n=0}^{N} f_n(t)\frac{T_n(x)}{\sqrt{1-x^2}}, \tag{5.32}$$

Using the main theorem 4, the solution can be obtained as follows:

Case (i-a): For $m=0, n=0$, we have

$$[\mu + 2\bar{\lambda}(ln2-d)+2]a_0(t) - \lambda\int_0^t a_0(\tau)H(|t-\tau|)d\tau = \frac{f_0(t)}{q-1}. \tag{5.33}$$

Case (i-b): For $m=0, n\geqslant 1$, and with the aid of the following formulas (Bateman and Ergeyli, 1985; Gradshteyn and Ryzhik, 1994)

$$\int_{-1}^{1}\frac{T_n(y)T_m(y)}{\sqrt{1-y^2}}dy = \begin{cases} 0, & n\neq m \\ \frac{\pi}{2}, & n=m\neq 0 \\ \pi, & n=m, \end{cases} \tag{5.34}$$

$$T_m(x)T_n(x) = \frac{1}{2}\left[T_{m+n}(x)+T_{|m-n|}(x)\right], \quad (m,n\geqslant 0), \tag{5.35}$$

$$\int_{-1}^{1} T_n(x)dx = \begin{cases} \frac{2}{1-n^2}, & n=0,2,4,\ldots \\ 0, & n=1,3,5,\ldots, \end{cases} \tag{5.36}$$

we get

$$\left[\bar{\mu} + \sum_{l=1}^{N}\left(2\frac{\bar{\lambda}}{n}A_{n,l}+2A_l^*\right)\right]a_n(t) - \lambda\int_0^t H(|t-\tau|)a_n(\tau)d\tau = \frac{f_n(t)}{q-1}, \quad (n\geqslant 1), \tag{5.37}$$

where

$$A_{n,l} = \begin{cases} \frac{1}{1-(n+l)^2}+\frac{1}{1-|n-l|^2}, & n+l = even \\ 0, & n+l = odd \end{cases} \tag{5.38}$$

and

$$A_l^* = \begin{cases} \frac{1}{1-l^2}, & l = even \\ 0, & l = odd. \end{cases} \tag{5.39}$$

Case (ii-c): For $m\geqslant 1, n=0$, we deduce

$$\left[\bar{\mu} + \sum_{m=1}^{M}\left(\frac{2\bar{\lambda}(1-2m^2)}{m(1-4m^2)}+2A_m^*\right)\right]a_0(t) - \lambda\int_0^t H(|t-\tau|)a_0(\tau)d\tau = \frac{f_0(t)}{q-1}. \tag{5.40}$$

Case (ii-d): For $|n-m|\geqslant 1, n\geqslant 1, m\geqslant 1, n\neq m$, we can establish

$$\left[\bar{\mu}+\sum_{m=1}^{M}\sum_{l=1}^{N}\left(\frac{\bar{\lambda}}{2|n-m|}B_{n,m,l}+2B^{*}_{m,l}\right)\right]a_n(t)$$
$$-\lambda\int_0^t H(|t-\tau|)a_n(\tau)d\tau=\frac{f_n(t)}{q-1}, \tag{5.41}$$

where

$$B_{n,m,l}=\begin{cases}\frac{1}{1-(n-m+l)^2}+\frac{1}{1-(n-m-l)^2}, & n-m+l=even\\ 0, & n-m+l=odd\end{cases} \tag{5.42}$$

and

$$B^{*}_{m,l}=\begin{cases}\frac{1}{1-(m+l)^2}+\frac{1}{1-(m-l)^2}, & m+l=even\\ 0, & m+l=odd.\end{cases} \tag{5.43}$$

Case (ii-e): For $(n+m)\geqslant 2, n\geqslant 1, m\geqslant 1$, we get

$$\left[\bar{\mu}+\sum_{m=1}^{M}\sum_{l=1}^{N}\left(\frac{\bar{\lambda}}{2|n-m|}Q_{n,m,l}+2B^{*}_{m,l}\right)\right]a_n(t)$$
$$-\lambda\int_0^t H(|t-\tau|)a_n(\tau)d\tau$$
$$=\frac{f_n(t)}{q-1}, \tag{5.44}$$

where

$$Q_{n,m,l}=\begin{cases}\frac{1}{1-(n+m+l)^2}+\frac{1}{1-(n+m-l)^2}, & n-m+l=even\\ 0, & n-m+l=odd.\end{cases} \tag{5.45}$$

The formulas (5.33)–(5.44) represent VIEs of the second kind. Many different methods, analytic or numeric, can be used to obtain the solution of VIE (Linz, 1985).

6. Numerical results

To obtain the numerical solution of F–VIE (5.32), we divide the interval time $[0,T]$ of VIEs (5.33)–(5.36),(5.37)–(5.40), (5.41)–(5.44) to obtain a LAS (Delves and Mohamed, 1985). Then, with the aid of the results of main theorem, we can calculate the unknown function $\Phi_N(x,t), -1\leqslant x\leqslant 1, 0<t<T<1$ for different times $t=\{0.0004, 0.03, 0.8\}$, when $N=60, M=7, V(|t-\tau|)=t^2-\tau^2, f(x,t)=x^2t^2, d=0.01, \lambda=0.1, q=10$ and $\mu=1$. The tables are given for different cases.

7. Conclusion

From the numerical results of Tables 1–3, we can deduce that; the error takes maximum value at the endpoints while takes the minimum when $x\approx 0$, and it becomes smaller when decreasing the time. By increasing N, the error decreases where the maximum error becomes 7×10^{-5}, at $N=100$ and the other parameters at $t=0.8$ are constants, while at $N=60$, the maximum error is 2×10^{-3}. The smooth function $p(x,y)$ has an effect for the potential function $\Phi(x,t)$, that is the error becomes smaller for bigger powers of x and y in $p(x,y)$, i.e. when $M=\{7,14\}$ and the other parameters at $t=0.8$ are constants, the maximum error becomes 2×10^{-3} and 2×10^{-4}, respectively. When q is big enough, the solution becomes more stable, for example, $q=1000, \lambda=10$ at $t=0.8$ the maximum error becomes 1.23×10^{-6}.

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