Models with Transaction Cost for Pricing European Options and Hedging

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Abstract: In this paper models with transaction costs for pricing of European options using mixed fractional Brownian motion (fbm) and Partial differential equation (PDE) model are considered. Investigation on price sensitivity to volatility and formulation of asymptotic strategy for replicating self financing assets are also made. Simulation experiments for the models are run to obtain the call and put prices using fbm with Hurst parameter $H = \frac{1}{3}$ and the Crank-Nicolson method for the numerical solution to the PDE model. It is found that stock prices increase with time to maturity dates, whereas, the call prices decrease steadily as time approaches maturity dates in conformity with theta hedging strategy.

Keywords: Simulation, European option, fractional Brownian motion, stochastic volatility.

Dedicated to evergreen memory of Late Professor M. S. Sasey

Author's contributions

In the recent times, development of financial instruments for pricing options are on increasing and the use of mixed fractional Brownian motion (fbm) for simulation of derivatives are not too common. In this paper, simulation experiments are designed and implemented using codes in MATLAB for the models considered. We obtained the call and put prices using fbm with Hurst parameter $H = \frac{1}{3}$ and the Crank-Nicolson method for the numerical solution to the PDE model using fbm realization. Investigation on price sensitivity to volatility and formulation of asymptotic strategy for replicating self financing assets are also made.

The author declares that he has no competing interests

1. INTRODUCTION

The Black-Scholes Model (BSM) is the most popular model for pricing financial derivatives. The model has been found to be inconsistent with empirical features of financial return series such as non independence, nonlinearity, etc. ([1], [2], [3], [4], [5] & [6]). The crux of assumption for deriving the Black-Scholes Model (BSM) was that the market is complete, that is, investors do not incur transaction cost in the trading with a risk-free asset (bond with constant return) and a risky asset (stock).Moreover, the price is governed by a geometric Brownian motion with constant rate of return and constant volatility ([7], [8] & [9]]).

Furthermore, using the BSM the value of an option, in the absence of arbitrary opportunity, is the expectation of discounted payoff at maturity time with risk-neutral measure in which the stock rate of return equals the risk-free rate. The question most researchers often asked is whether it is possible to replicate an option with transaction cost without involving some risks ? In some trading in options element of risks have been found to be essential (see Figlewski [10]).

Transaction costs in organized markets are often assumed to be absent, but in the over the counter trades (OTC) for pricing of exotic options, transaction costs are involved in the trading. BSM which forms the basis of most financial calculators no arbitrage opportunity and no transaction cost are the fundamental assumptions for the trading. In practice, a transaction cost needs to be taken into consideration for accurate for pricing of options especially European call and put options. Figlevski ([10]) demonstrated that even in discrete-time model, transaction cost is a substantial factor in hedging.

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There are three general methods for calculating transaction costs. A transaction cost: i) is calculated as a percentage of the underlying security; ii) is a fixed charge for each shares; or iii) is a single flat fee charged regardless of the number of shares. Transaction costs are sometimes calculated as combination of these three methods ([1], [11], [12], [13] & [14]).

We will not look at ways the transaction costs are charged, but will focus on models with transactions cost as a percentage of the underlying security. In a finite state market, the fundamental theorem of arbitrage has two parts. The first part relates to existence of a risk neutral measure, while the second relates to the uniqueness of the measure (See [15], [16] & [17]):

1) The first part states that there is no arbitrage if and only if there exists a risk neutral measure that is equivalent to the original probability measure.
2) The second part states that a market is complete if and only if there is a unique risk neutral measure that is equivalent to the original probability measure.

The fundamental theorem of pricing is a way for the concept of arbitrage to be converted to a question about whether or not a risk neutral measure exists.

The fundamental theorem in more general markets when stock price returns follow a single Brownian motion, there is a unique risk neutral measure. When the stock price process is assumed to follow a more general semi martingale process (see [15], [18]), then the concept of arbitrage opportunity is too strong, and a weaker concept with vanishing risk must be used to describe this opportunity in an infinite dimensional setting.

There are several interesting models for pricing and hedging of European options in literature we will only consider some selected few ones relevant to our study. We note that, Leland [18] used relaxation effect to derive an option price that is equivalent to BSM but with modified volatility $\hat{\sigma} = \sigma\left(1 + \sqrt{\frac{2}{\pi}\frac{\varphi}{\sigma\sqrt{\Delta t}}}\right)^{\frac{1}{2}}$ .where σ is the original volatility, φ is the proportional transaction cost and Δt is the transaction frequency. In this formula, both φ and Δt are assumed to be small while keeping the ratio $\varphi/\Delta t$ at order of one.

We note that the replication of a European option with transaction cost was proposed by Xiao et. al. [6]; Lai and Lim [19] in the limiting $t \to \infty$, showed that the replication price approaches the super replication price of the option plus the liquidation value of the initial endowment (see [6] & [20]).

Zakamoaline ([13] & [21]) considered utility based option pricing and hedging problem in a market with both fixed and proportional transaction costs. Simulation comprising of the performance of the utility based hedging strategy against the asymptotic strategy was made.

Furthermore, the problem of delta-hedging portfolio of options with transaction costs was considered by Les and Steward ([22]) and results obtained by maximizing expected utility (or minimizing a cost function on the replication error).

David ([2]) gave an in-depth review on the use of fractional Brownian motion for pricing European options and Delbaen and Schachermayer [23] studied the European option pricing with transaction cost by the use of stochastic volatility and obtained results on replication of options using asymptotic analysis.

Many financial instruments are being developed in the recent times for pricing options. Therefore, this paper considers models with transaction costs which are driven by the mixed fractional Brownian motion which may be used for pricing European exotic options.

Simulation experiments would be run for hedging the cost for a self-financing trading strategy and replicating a portfolio using fractional binomial motion(fbm) would be made ([24]&[25]). We will also consider a trading strategy using a PDE model with transaction cost and make use of the Crack-Nicolson scheme to derive numerical solution to the PDE model.

Simulation experiments would be carried out to obtain the call and put prices for European option using the fbm with Hurst parameter $H = \frac{1}{3}$. Moreover, the modified BSM of Leland type with variable volatility and the transaction cost will be used to price the European option couple with the investigation on price sensitivity to volatility. Finally, the asymptotic strategy for replicating self financing assets will also be considered.

2. PRELIMINARY DEFINITIONS AND NOTATIONS

Notation

S_{max}	boundary value for asset with spot price $S(t)$
σ	constant volatility
$\bar{\sigma}$	variable volatility
sgn(Γ)	transaction cost
σ_H	fbm volatility
k	index number
δt	discrete step of time t
$S0$	initial asset price
K	strike price
r	risk free interest rate
q	Dividend
δx	discrete step of space x
	range of x such that $0 \le x \le 0.15$, using step size 0.01

$x\ :\ 0\ :\ 0.01\ :\ 0.15$

$[C, P] = BSA1$ Matlab Code for computing call and put prices

Remark 1

We will make use of the denotation $BSA1$ to invoke the code in the Matlab library for computing European option price using the BSM in the Matlab Financial Tool Box.

$BSA1 = blsprice(S0, K, r, T, sigma, q); sigma = \sigma;$

That is $BSB1 = blsprice(S0, K, r, T, sigma, q); sigma = \bar{\sigma},$ this does same computation as $BSA1$ except that the

constant volatility has been replaced by variable volatility, i.e. $\bar{\sigma}$.

If we introduce a counter x into $BSB1$ we can compute prices with respect to changes in volatility by making use of:

$BSA2 = blsprice(S0, K, r, T, sigma + x, q);$

Here $sigma = \sigma$ and x is the counter to be used.

$BSB2 = blsprice(S0, K, r, T, sigma + x, q);$

We will take $sigma = \bar{\sigma}$ when pricing with variable volatility.

$|\Gamma|$ is the transaction cost.

Definition 1

Measure to reduce or eliminate an exposure to risk is referred to as hedging. The term hedging an exposure and hedging a risk are used interchangeably.

Definition 2 *[26]*

Liquid Assets are assets which are themselves money, or can be converted into money with minimum delay and risk of loss .Short -dated marketable security e.g. treasury bills .**Liquidity** is the property of having liquid asset.

Definition 3

Let $h_i(t)$ denotes the number of stocks with number i in the portfolio at time t and $S_i(t)$ the price of i

stock in a frictionless market with continuous time $v(t) = \sum_{i=1}^{n} h_i(t) S_i(t)$ then the portfolio is said to be self-financing if

$dv(t) = \sum_{i=1}^{n} h_i(t) dS_i(t)$ is it replication and they have same cash flow (Static) or dynamic replication with different cash

flow but same Greeks .A market is complete if and only if there is a unique martingale measure to model the behavior of the asset.

3. STATEMENT OF PROBLEM

3.1 Fractional Brownian motion

Fractional Brownian motion (fbm) was introduced by Kolmogrov [27] and studied extensively by Mandelbrot and Van Ness ([25]).

A Gaussian process $B^H = \{B_t^H, t \geq 0\}$ is called fractional Brownian motion (fbm) with Hurst parameter $H \in (0,1)$ if $E\{B_t^H\} = 0, E\{B_t^H, B_s^H\} = R_t(t,s) = \frac{1}{2}s^{2H} + t^{2H} - |t-s|^{2H}$. B_t^H can be obtained from integral representation as (see [2])

$$B_t^H = \frac{1}{C(H)} \int_0^\infty [(t-s)^+)^{H-\frac{1}{2}} - ((-s)^+)^{H-\frac{1}{2}}]dw_s \tag{1}$$

Where $\{w(A) \text{ is a borel sunset }\}$ of R and it is a Brownian measure and

$$C(H) = \sqrt{\int_0^\infty ((1-s)^+)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}})^2 ds + \frac{1}{2H}} \tag{2}$$

The equation (1) was obtained by Mandelbrot and Van Ness in [25].If $H = \frac{1}{2}$ then $R_{\frac{1}{2}}(t,s) = \min(s,t)$, therefore B_t^H is the ordinary Brownian motion; generally,

$$E\left((B_t^H - B_s^H)^{2k}\right) = \frac{(2k)!}{k!2^k}|t-s|^{2k} \tag{3}$$

Fbm is said to have stationary increment in the interval $[s,t]$ if it has normal distribution with $E\{B_t^H\} = 0$ and $E\left((B_t^H - B_s^H)^2\right) = |t-s|^{2H}$.

Remark 2

B_t^H is generally a self similar process in which ,for any $c > 0, B_{ct}^H = c^H B_t^H$ for $t \in R. B_t^H$ is stationary since it has the property that $B_{t+s}^H - B_t^H \sim \delta^H$ implies $\sum_{j=1}^{2n} |B_{j2^{-n}}^H - B_{(j-1)2^{-n}}^H| \sim (2^n)^{1-pH}$ as $n \to \infty$ .The order of variation of B_t^H is in fact $p < H^{-1}$ and zero if $p > H^{-1}$ .This is only consistent with semi -martingale behavior if $H = \frac{1}{2}$ for more study on fractional Brownian motion (fbm)($\bar{\sigma}$ see[2],[6],[24]&[28]).

The mean self financing delta hedging strategy in a discrete-time setting was considered by Leland [18] by the use of the fundamental equation of asset with variable volatility .We will consider the continuous version wherein we will allow $\delta S \to 0$. Hence we have a continuous stochastic differential of the form

$$\frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} + \frac{\bar{\sigma}^2}{2} S_t^2 \frac{\partial^2 C}{\partial S_t^2} - rC = 0 \tag{4}$$

Where the modified volatility (see [18]) is given by

$$\bar{\sigma} = \sqrt{\sigma^2 + \delta_H^2(\delta t)^{2H-1} + k\sqrt{\frac{2}{\pi}}(\frac{\sigma^2}{\delta t} + \sigma_H^2(\delta t)^{2H-2})\text{sgn}(\Gamma)} \tag{5}$$

$$C(t,S_t) = E^Q(e^{-rT} \max(S_T - K, 0)) \tag{6}$$

$$= S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{7}$$

And $\Gamma =| \frac{\partial^2 C}{\partial S_t^2} |$ is the transaction cost .If $\Gamma < 0$ and $\delta t \to 0$ then the problem is ill-poised, but Γ is always positive for the simple European option with no transaction cost. When there is transaction cost, the model becomes the standard BSM with modified volatility.

In the equation (7) is volatility of the market using fbm method and it includes the transaction cost. When $H = \frac{1}{2}$

and $\Gamma = 0$, the modified volatility $\bar{\sigma} = \sigma$, that is, the model becomes the BSM.

The European option call price under this dispensation is

Where $d_1 = \frac{\ln(\frac{S_t}{K})+(r+\frac{\bar{\sigma}^2}{2})(T-t)}{\bar{\sigma}^2 \sqrt{T-t}}, d_2 = d_1 - \bar{\sigma}^2 \sqrt{T-t}$

And $\bar{\sigma} = \sqrt{\sigma^2 + \delta_H^2 (\delta t)^{2H-1} + k \sqrt{\frac{2}{\pi}(\frac{\sigma^2}{\delta t} + \sigma_H^2 (\delta t)^{2H-2}) \operatorname{sgn}(\Gamma)}}$.

$N(.) \sim N(0,1)$.

Xiao et.al. [6] Showed that the minimum price of an option with transaction cost is
Such that

$$C_{\min} = \min C(t, S_t) \tag{8}$$

$$\text{Subject to } C(T, S_T) = \max(S_T - K, 0) \tag{9}$$

$$C(t, S_t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2) \tag{10}$$

Stochastic calculus with respect to Wick integral was used to derive the fbm analogue of the BSM (see [2]) as

Where $d_{1,2} = \frac{\ln(\frac{S_t}{K})+(r(T-t)\pm\frac{\bar{\sigma}^2}{2}(T^{2H}-t^{2H}))}{\bar{\sigma}^2 \sqrt{T^{2H}-t^{2H}}}$.

3.2 Replication of portfolio with self financing

Let us consider an European call option in which an investor invested in stock with strike price K and maturity at T, with mean risk of return $r > 0$ and volatility $\sigma > 0$. By BSM we have the following stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dw_t \tag{11}$$

Where $\{w_t, t \geq 0\}$ is a standard Brownian motion with no transaction cost

Let $(x_t)_{t \geq 0}$ be a continuous process which admits the following decomposition

$$x_t = L_t - M_t \tag{12}$$

Where L_t is increasing in t and it is a deterministic process and M_t is martingale process.

Now consider the portfolio with transaction cost such that (see [29])

$$dx_t = dL_t - dM_t \tag{13}$$

$$dy_t = ry_t dt - aS_t dL_t \tag{14}$$

Where

x_t Is the number of share held in stock, y_t is the number of share held in bonds and $r > 0$ is a fixed risk-free rate with $a = 1 + \lambda$ and $b = 1 - \mu; \mu, \lambda \in [0,1]$, λ is the cost the investor pays in the purchase of the underlying stock while μ is paid for the sale of the underlying asset.

Let

$$\Psi(L, M, t, T) = a \int_t^T e^{r(T-u)} S_u dL_u - b \int_t^T e^{r(T-u)} S_u dM_u$$

Such that

$$C^i(L, M, t, T) = \Psi(L, M, t, T) - z^i(S_T, X_T), i = 1, 2, 3 \tag{15}$$

Therefore, $C^i(L, M, t, T)$ is the total cost with the transaction cost incur over $[t, T]$ and $z^i(S_T, X_T)$ is the liquidity value of the stock for the share at maturity S_T.

The terminal wealth of the investor is

$$\Omega_T^i = y_t + z^i(S_T, x_T) = y_t e^{r(T-t)} - C^i(L, M, t, T) \tag{16}$$

Where y_t denotes the initial wealth at time t . The hedging cost is the difference between the terminal value of the initial wealth and the terminal wealth.

3.3 Replicating a self- finance portfolio with transaction cost

We will replicate a European option with transaction cost using strategies proposed by Xiao et. al. [28].

Consider a portfolio at the current time t as follows

$$\Pi_t = X_1(t)S_t + X_2(t)D_t \tag{17}$$

Where $X_1(t)$ is the number of shares bought at time t, $X_2(t)$ is the number of bonds bought at time t, S_t and D_t are the shares and amount spent on the bonds at time t respectively.

3.4 Numerical solutions

Partial differential equations in continuous form can be approximated by discrete finite difference scheme (FDS).FDS of option pricing are numerical methods used in Mathematics of Finance for the valuation of option. We will make use of FDS to transform the continuous PDE into discrete algebraic equation and then solve the equation to obtain the approximate solution to the PDE.

We discretize the continuous data (S, t) by discrete data (S_i, t_j) such that:

$$S = 0, \delta S, 2\delta S, ..., M \delta S = S_{max}$$

$$t = 0, \delta t, 2\delta t, ..., N \delta t = T$$

Therefore, $\delta S = \frac{S_{max}}{M}, \delta t = \frac{T}{N}$ and $C_{i,j} = C(i\delta S, j\delta t) \rightarrow$ the grid nodal points. In the nodal diagram in the Fig.1 (i, j) represents the computation at $(i\delta S, j\delta t), i = 0, 1, ..., M, j = 0, 1, ..., N$.

Boundary conditions

For Call option $C(S, T) = \max[S(T) - K, 0]$

For very large $S(T), C(S_{max}, t) = 0.$

The PDE is discretized at each value at each lattice point is specified as a function of value at later and adjacent point see the nodal diagram.
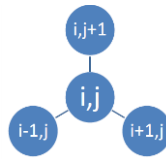

Figure 1: Nodal diagram

3.5 Cranck-Nilcolson method

Crank-Nicolson method (CNM) is based on central difference scheme in space and the trapezoidal rule in time with second-order convergence with time. The choice of CNM for our study is because it is efficient, it saves CPU time, has less absolute error and it is unconditionally stable. The errors of approximation for CNM is of order $\sim o((\delta\tau)^2 + (\delta x)^2)$ where $\delta\tau$ and δx are time and space discretization respectively. The CNM we will consider will include the transaction cost in the variable volatility function in the equation (7). The model can also be studied by including the transaction cost in the PDE but the volatility must be constant.

If we apply Cranck-Nilcoson method to the PDE above we have

$$\frac{\partial^2 C(S, t_{j+\frac{1}{2}})}{\partial t^2} = \frac{C(S, t_{j+1}) - C(S, t_j)}{\delta t} + o(\delta^2 S) \tag{18}$$

We will make use of

$$\frac{\partial^2 S(x_i, t_{j+\frac{1}{2}})}{\partial x^2} = \frac{1}{2}\left[\frac{\partial^2 S(x_i, t_{j+1})}{\partial t^2} + \frac{\partial^2 S(x_i, t_j)}{\partial t^2}\right] + o(\delta S^2) \tag{19}$$

By the central approximation for differentiation with respect to t in $C(S_i, t_{j+\frac{1}{2}})$ we have

$$\frac{\partial C(S, t_{j+\frac{1}{2}})}{\partial t} = \frac{C(S, t_{j+1}) - C(S, t_j)}{\delta t} + o(\delta^2 t) \tag{20}$$

$$C(0,t) = Ke^{-r(T-t)}, C_{i,N} = \max[K - i\,\delta S, 0]$$

$$C_{0,j} = Ke^{-r(N-j)\delta t}, j = 0,1,2,...,N$$

$$C_{M,j} = 0, j = 0,1,...,N$$

4. RESULTS

The solution to the equation(1) by the use of Ito formula which is a standard result in the literature is $S_t = S_0\exp(rt - \frac{\sigma^2}{2}t + \sigma w(t))$ where   is the ordinary Brownian motion, r is risk free interest rate and σ is the constant volatility([33]).

Let us replicate a portfolio using fbm realization then the solution to the equation (11) is

$$S_t = S_0\exp(rt + \sigma B_t^H - \frac{\sigma^2}{2}t^{2H}) \tag{21}$$

Where B_t^H is fmb with Hurst parameter H , r is the mean rate of return and $\sigma > 0$ is the constant volatility .If $H = \frac{1}{2}$ we note that the original BSM can be recovered from the fmb version. The spot price of an asset using fbm form is, therefore, given as $S_t = S_0\exp(rt + \sigma B_t^H - \frac{\sigma^2}{2}t^{2H})$. Where S_0 the initial stock price is ,K is the strike price ,r is risk free interest rate ,T is the expiration time or the maturity time, $sigma = \sigma$ is the volatility of the market and Γ is the transaction cost.

Back to the equation (14), the stochastic differential equations with initial wealth at time t using the fmb realization is given as

$$dy_t = ry_t dt + aS_0\exp(rt + \sigma B_t^H - \frac{\sigma^2}{2}t^{2H})dL_t \tag{22}$$

And the terminal wealth of the investment can be calculated to be

$$\Omega_T^i = y_t e^{r(T-t)} - a\int_t^T e^{r(T-u)}S_u dL_u - b\int_t^T e^{r(T-u)}S_u dM_u$$
$$- z^i(S_T, X_T) \tag{23}$$

The European option can be price with transaction cost ($\text{sgn}(\Gamma) \neq 0$) or with no transaction cost (i.e. $\text{sgn}(\Gamma) = 0$).

The simulation experiments carried out in this paper are done using Matlab codes. We start our simulation with the Black Schole Model (BSM) for $S_0 = 30$, take $S_{k-1} = S_0 + \Delta S_k, \Delta S_k = 1, k = 40, K = 50, r = 0.1$ and $\sigma = 0.02$. In the Fig.2a we have the plot for .Then we have the plot in the Fig.2b below:
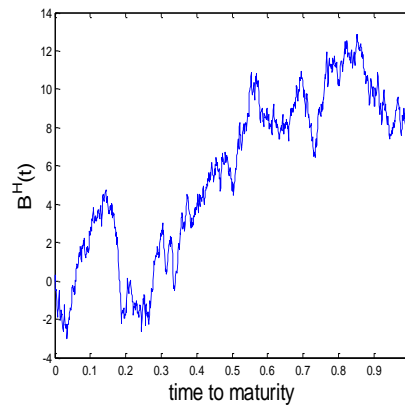
Figure 2a: Fractional Brownian motion $H = \dfrac{1}{3}$

Fig.2b shows how Stock price changes as time approaches maturity times .Fig.3 is the simulation of stock price using the fbm
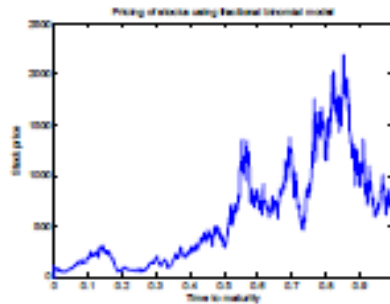

Figure 3: Stock price using binomial motion

Let us consider replicating a portfolio using fbm realization then solution to the equation (1) (see [2]) is $S_t = S_0 \exp(rt + \sigma B_t^H - \dfrac{\sigma^2}{2} t^{2H})$ where B_t^H is fmb with Hurst parameter H , r is the mean rate of return and $\sigma > 0$ is the volatility .If $H = \frac{1}{2}$ we note that we recover the original BSM from fmb ([14]). The plot of graph of $S_t = S_0 \exp(rt + \sigma B_t^H - \frac{\sigma^2}{2} t^{2H})$ using $H = 0.33, \sigma = 0.24, S0 = 100, r = 0.02$ and $N = 1000$ is in the Fig.4. We observe also that stock price by the use of fbm realization increases with time to maturity time. Moreover, the call price decreases steadily as time approaches maturity in conformity with theta hedging strategy (i.e. the theta must always negative).

Let $K = 90$ naira and $r = 0.02$ then the call price of the stock using fbm as time approaches maturity is shown in Fig.4.
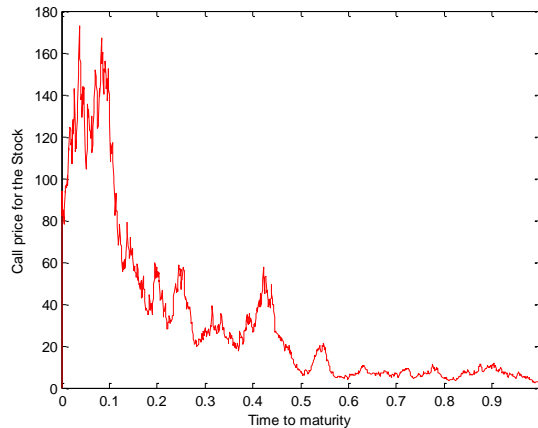
Figure 4: Call price for stock using fractional binomial model

Let us consider a portfolio such that

$$\Pi_t = x_t e^{rt} + y_t S_t \tag{24}$$

Where x_t and y_t are stochastic processes and x_t is describing the number of stocks owned by the investor at time t and y_t is the number of bonds owned by the investor at time t.

The portfolio is self-financing if

$$\Pi_t = \Pi_0 + r \int_0^t x_s e^{rs} ds + \int_0^t y_s S_s ds \tag{25}$$

This means that there is no fresh investment and there is no consumption if $R_t(t,s) \geq 0$, that is $\sum_{i,j=1} a_i a_j R_t(t_i, s_j) \geq 0$ for any sequence of real-numbers $\{a_i\}, i = 1, 2, 3, ..., n$. By definition, an arbitrage is a self-finance portfolio which satisfies $\Pi_0 = 0, \Pi_T \geq 0$ and $P(\Pi_T > 0)$. Fig.5 is the call prices as time approaches maturity times for various values of σ. Other parameters used in the simulation in the Fig. 5 are the same as in the Fig.4 except σ is changing.
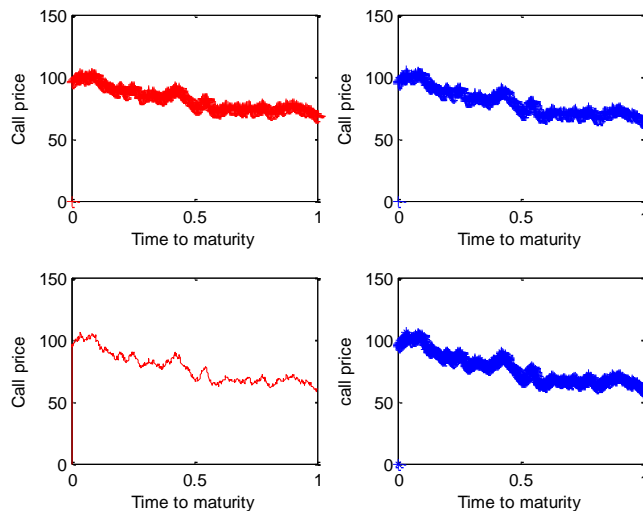

Figure 5: Call price as time approach maturity times

Fig.5a, $\sigma = 0.025$, Fig.5b , $\sigma = 0.030$, Fig5.c , $\sigma = 0.035$ and Fig.5.d , $\sigma = 0.035$.
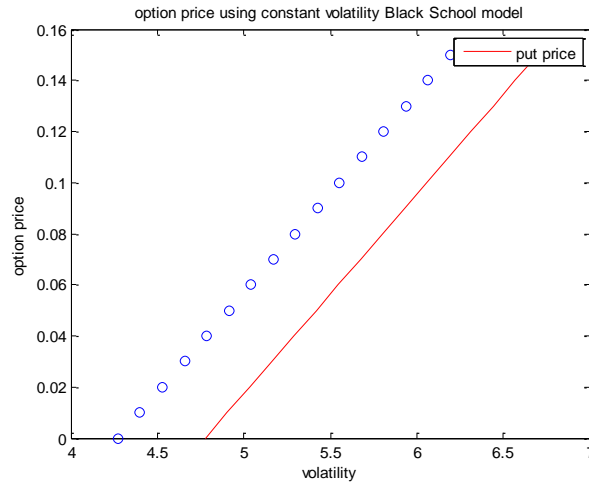

Figure 6: Option prices using constant volatility Black Schole model

We have simulated the BSM with modified volatility using the Matlab code $BSB2$ and $BSA2$. Therefore, for $\sigma = 0.350, r = 0.070, K = 52, S_0 = 50, T = 5/12, q = 0, \bar{\sigma} = 0.625,$

$H = 0.333, \sigma_H = 0.667$ and $\text{sgn}(\Gamma) = 0.500$ the call price and put prices were found to be $C = 4.2715$ and $P = 4.7768$ respectively. Using increment of $x = 0.01$ and volatility lying between 0 and 0.15 we have the graphs of stock price against volatility in the Fig.6, the blue curve is the call prices while the red star curve is the put prices. The Fig. 8 gives the graph of option price against volatility.
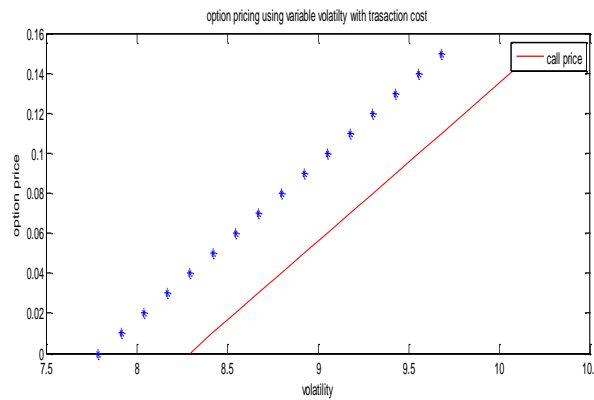

Figure 7: Graph of option price against volatility

Avellaneda and Paras considers the nonlinear function $\hat{Q} = \sqrt{\frac{2}{\pi}} \frac{\phi}{\sigma \sqrt{\Delta t}} \frac{A}{|A|}, \Delta t$ is transaction frequency where σ original volatility and ϕ transaction cost.
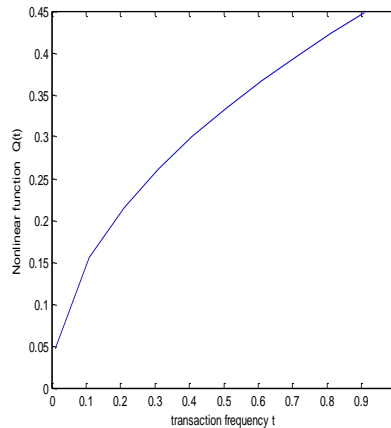
Figure 8: The function Q (t) with transaction frequency

The Fig.9 gives corresponding graphs using variable volatility model with transaction cost using the Avellaneda and Paras function $Q(t)$ when $\phi = 0.2, \sigma = 0.34$ and $0.01 \le \Delta t \le 1, stepsize = 0.1$. we observe that $Q(t)$ is a concave and increasing function with respect to transaction frequency. Avellaneda and Paras function gives a clue on how develop a modified volatility function which should be concave and increasing with transaction frequency .We can easily check that $\bar{\sigma}$ has the two properties.

Now, let us study how variability in stock will affects portfolio using asymptotic strategy. First of let us assume that $|\delta B_t| = o((\delta t)^{3/2} \sqrt{\sigma_t^H})$ ([6]&[13]) which will then implies that $\frac{\delta B_H}{\delta B_t} \to 0$ as $\delta t \to \infty$. Base on the assumption on $|\delta B_t|$ we will derive asymptotic formula for S_t from variability on portfolio $\delta\Pi_t$ using mixed fmb and the modified volatility. It is not difficult to show that

$$\delta S_t = rS_t \delta t + S_t(\sigma \delta B_t + \sigma_H \delta B_t^H)$$
$$+ \frac{S_t^2}{2}(\sigma \delta B_t + \sigma_H \delta B_t^H)^2 + o((\delta t)^{\frac{3}{2}} \sqrt{(\log(\delta t))^{-1}}) \tag{26}$$

Therefore

$$\frac{S_t^2}{2}(\sigma \delta B_t + \sigma_H \delta B_t^H)^2 + o((\delta t)^{\frac{3}{2}} \sqrt{(\log(\delta t))^{-1}} \tag{27}$$

Therefore the change in the value of the portfolio is

$$\delta\Pi_t = X_1(t)\delta S_t + X_2(t)\delta D_t - \frac{k}{2}|\delta X_1(t)|S_t \tag{28}$$

Where the transaction costs of the rehedging over the interval is equal to $\frac{k}{2}|\delta X_1(t)|S_t$ .Therefore hedging order of $|\delta X_1(t)|$ is

$$|\delta X_1(t)| - S_t|\frac{\partial X_1(t)}{\partial S_t}|(\sigma \delta B_t + \sigma_H \delta B_t^H)\|\frac{\partial X_1(t)}{\partial S_t}| + o(\delta t) \tag{29}$$

Hence equation (28) becomes

$$\delta\Pi_t = X_1(t)\delta S_t + rX_2(t)dD_t \delta t + |\delta X_1(t)|$$
$$- \frac{k}{2}\delta_t^2(\sigma \delta B_t + \sigma_H \delta B_t^H)|\delta X_1(t)|S_t + o(\delta t) \tag{30}$$

Thus as $\delta t \to 0$ it implies that

$$d\Pi_t = X_1(t)dS_t + rX_2(t)\delta D_t dt + |\frac{dX_1(t)}{dS_t}|$$

$$-\frac{k}{2}\delta_t^2(\sigma\delta B_t + \sigma_H \delta B_t^H)|\frac{dX_1(t)}{dS_t}|S_t \tag{31}$$

But $dS_t = rS_t dt + \bar{\sigma}\delta B_t^H$ hence

$$d\Pi_t = X_1(t)(rS_t dt + \bar{\sigma}\delta B_t^H) + rX_2(t)\delta D_t dt + |\frac{dX_1(t)}{dS_t}|$$

$$-\frac{k}{2}\delta_t^2(\sigma\delta B_t + \sigma_H \delta B_t^H)|\frac{dX_1(t)}{dS_t}|S_t \tag{32}$$

Therefore

$$(\delta t)^2 = S_t^2(\sigma\delta B_t + \sigma_H \delta B_t^H)^2 + o(((\delta t)^{\frac{3}{2}}\sqrt{(\log(\delta t))^{-1}})$$

Thus

$$S_t = \left[\left(\sigma\delta B_t + \sigma_H \delta B_t^H\right)^{-2}\delta^2 t + o\left(\frac{(\delta t)^{3/2}\sqrt{\log|((\delta t)-1)|}}{\left(\sigma\delta B_t + \sigma_H \delta B_t^H\right)^2}\right)\right]^{1/2}$$, Valid for $\delta t \geq e = 2.71$. Using the asymptotic formula for

S_t above, for $\delta = 0.350, \sigma_H = 0.667, \delta t = 2.80, \delta B_t = 0.02$ and $\delta B_t^H = 0.015$, we get $S_t = 164.7$.

Now let us consider the nonlinear BSM in the partial differential equation (PDE) form .In order to have a numerical stable solution to the model, the Crank-Nicolson method was made use of. Moreover, equations (4-6) were used to convert the PDE into matrix form as follows:

$$AF_i^j + b = 0 \tag{33}$$

Where

$$A = \begin{bmatrix} r - \sigma^2 i^2\varpi & \frac{r-\sigma^2 i^2\varpi}{2} & 0 & ... & 0 & 0 \\ 0 & r - \sigma^2 i^2\varpi & \frac{-r+\sigma^2 i^2\varpi}{2} & 0 & ... & 0 \\ 0 & 0 & r - \sigma^2 i^2\varpi & \frac{-r+\sigma^2 i^2\varpi}{2} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & r - \sigma^2 i^2\varpi & \frac{-r+\sigma^2 i^2\varpi}{2} & 0 \\ 0 & 0 & ... & 0 & \frac{-r+\sigma^2 i^2\varpi}{2} & r - \sigma^2 i^2\varpi \end{bmatrix}$$

$$\varpi = [1 + e^{r(T-t)}a^i i^2(F_{i+1}^j - 2F_i^j + F_{i-1}^j)], b = [0,..., \frac{rN+N^2\sigma^2}{2}(s_{max} - Ke^{-r(T-r)})]$$

Where $a = \mu\sqrt{\gamma N}, \gamma$ is risk aversion factor, N is the number of option to be sold and μ is the proportion of transaction cost. Using the equation (33) the simulation to the PDE was obtained. For $\sigma = 0.350, H = 0.33, \sigma_H = 0.67, K = 50$, $\delta t = 2.8, \delta B_t = 0.02$ and the transaction cost= $\text{sgn}(\Gamma) = 0.500 = \mu$ using the variable volatility formula in the equation (22) we found that $\bar{\sigma} = 4.1767$. Using the values $r = 0.070, S_0 = 50, T = 5/12, S_{max} = 200, q = 0, N = 2$ and the CNM. The call price was found to be 41.073.

5.   CONCLUSIONS

The pricing of European options with transaction cost using mixed fractional Brownian motion and the PDE model have been considered. The Crank-Nicolson scheme was used to obtain numerical solutions to the PDE model by pricing the European options with transaction cost. Asymptotic formula for stock price is derived using the mixed fractional Brownian motion and simulations are made for the models at various scenarios.

The pricing of a European option with transactions cost can be treated as correction term in the BSM, as mentioned above, the BSM has limitation in describing the real market behaviour. Hence models with transactions cost and stochastic volatilities offer accurate models for pricing options. Modeling and simulation with transaction cost have several interesting aspects such as option pricing via utility maximization and pricing using mean-reverting stochastic volatility which this paper did not cover. Option pricing in the presence of transaction costs can also be based on an

optimal portfolio approach. Approximate replication schemes are noted to work well for small levels of transaction costs, but they cannot cope with larger values of transaction costs. Further research work can be extended to these aspects.

ACKNOWLEDGEMENTS

The author is grateful to the Plateaus State University, Bokkos, Nigeria and the anonymous referee whose suggestions helps to improve the content of the paper.

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