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On the Cartwright Power-of-Cosine Circular Distribution CPC(μ,ψ) - Some Distributional Properties and Characterizations

M. Shakil[1], M. Ahsanullah[2] and B.M. Golam Kibria[3]

[1]*Miami Dade College, Hialeah, FL, USA,*
[2]*Rider University, Lawrenceville, NJ, USA,*
[3]*Florida International University, Miami, FL, USA*

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Abstract: For modelling the directional spectra of ocean waves, Cartwright introduced a power-of-cosine circular distribution, (cf. Cartwright, D. E. (1963), "The use of directional spectra in studying the output of a wave recorder on a moving ship, In *Ocean Wave Spectra*", pages 203—218, Prentice Hall, New Jersey). Some distributional properties of the Cartwright's power-of-cosine circular distribution will be discussed in this paper. Based on these properties, some characterizations of this distribution will be given using the truncated moment, order statistics and record values.

Keywords: Cartwright distribution, Characterizations, Truncated Moments

1. Introduction

For an absolutely continuous circular random variable X, several circular distributions have been introduced by various authors and researchers; see, for examples, Mardia and Jupp [17], and Jammalamadaka and SenGupta [13], among others. A probability distribution is called a circular distribution if its total probability is concentrated on the circumference of a unit circle. Since each point on the circumference represents a direction, such a distribution is a way of assigning probabilities to different directions or defining a directional distribution. The range of a circular random variable, X, is measured in radians, and may be taken to be $0 \le x < 2\pi$ or $-\pi \le x < \pi$. For an absolutely continuous circular random variable X, (with respect to the Lebesgue measure on the circumference of the circle), a probability density function $f(x)$ exists and has the following basic properties: (i) $f(x) \ge 0$; (ii) $\int_0^{2\pi} f(x)\,dx = 1$ and (iii) $f(x) = f(x + 2\pi k)$, for any integer k (that is, $f(x)$ is periodic).

Cartwright [7] introduced a power-of-cosine circular distribution, $X \sim \mathrm{CPC}(\mu,\psi)$, for a continuous random variable X, to study the modelling of the directional spectra of ocean waves. The probability density function (pdf) of power-of-cosine circular distribution is given by

$$f_\psi(x) \propto \left[1 + \cos(x-\mu)\right]^{\frac{1}{\psi}}, \quad \mu - \pi < x \le \mu + \pi, \tag{1.1}$$

E-mail address: mshakil@mdc.edu, ahsan@rider.edu, kibriag@fiu.edu

where $0 \le \mu < 2\pi$ is the location parameter and $\psi\ (>0) \in \Re$. For details, on it, we refer our readers to Cartwright [7] and Jones and Pewsey [15], among others. If $c(\psi)$ denotes the normalizing constant, then integrating (1.1) with respect to x in the interval $\mu - \pi < x \le \mu + \pi$, it is easy to see that

$$C(\psi) = \frac{1}{\int_{\mu-\pi}^{\mu+\pi} \left[1 + \cos(x-\mu)\right]^{\frac{1}{\psi}} dx}$$

$$= \frac{2^{\left(\frac{2}{\psi}+1\right)} \left(\frac{2}{\psi}+1\right) B\left(\frac{1}{\psi}+1, \frac{1}{\psi}+1\right)}{\pi\, 2^{\left(\frac{1}{\psi}+2\right)}}, \quad \psi > 0, \tag{1.2}$$

which follows from the Equation 3.631.9 of Gradshteyn and Ryzhik [12]), where $B(.,.)$ denotes the beta function. Thus, using the definition of the beta function in terms of the gamma function $\Gamma(.)$, that is, $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, in Eq. (1.2), the expression for the normalizing constant, $C(\psi)$, reduces to the following:

$$C(\psi) = \frac{2^{\left(\frac{1}{\psi}-1\right)} \Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi \Gamma\left(\frac{2}{\psi}+1\right)}, \quad \psi > 0 .$$

Accordingly, the pdf (1.1) of Cartwright's distribution is given by

$$f_\psi(x) = \frac{2^{\left(\frac{2}{\psi}-1\right)} \Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi \Gamma\left(\frac{2}{\psi}+1\right)} \left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}, \quad \mu - \pi < x \le \mu + \pi . \tag{1.3}$$

The cumulative distribution (cdf) $F(x)$ corresponding to the probability density function (1.3) is given by

$$F(x) = \frac{2^{\left(\frac{2}{\psi}-1\right)} \Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi \Gamma\left(\frac{2}{\psi}+1\right)} \int_{\mu-\pi}^{x} \left[\cos\left(\frac{t-\mu}{2}\right)\right]^{\frac{2}{\psi}} dt, \quad \mu - \pi < x \le \mu + \pi , \tag{1.4}$$

which cannot be evaluated analytically in closed form, and hence should be solved numerically by some appropriate numerical quadrature rules, such as the Newton-Cotes or Gaussian quadrature formulas. For $\psi = 1$, Cartwright distribution reduces to the circular distribution introduced by Jeffreys [14]. Using Maple software, the plots of the pdf (1.3) for two different sets of values of the parameters, are presented in Figures 1 - 2. If we review these figures, we can see the effects of the parameters. It is noted that, since the Cartwright's distribution is an even function, the pdf of X ~ CPC (μ, ψ) is symmetric about its mean for any values of the parameters.

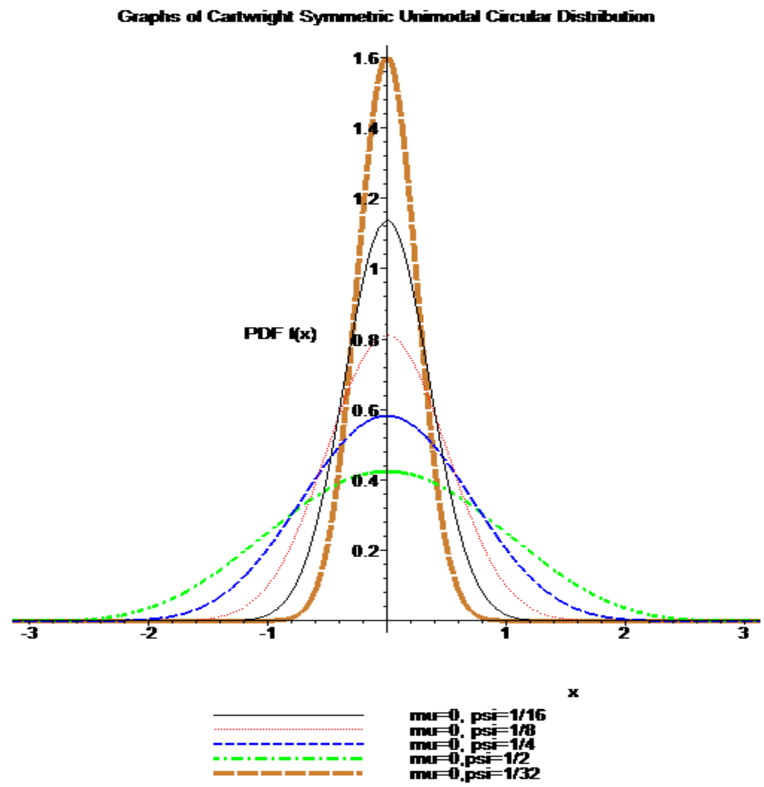

Figure 1. PDF $f(x)$ of $X \sim CPC(\mu, \psi)$, Eq. (1.3), when $\mu = 0$, and $\psi = \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$.

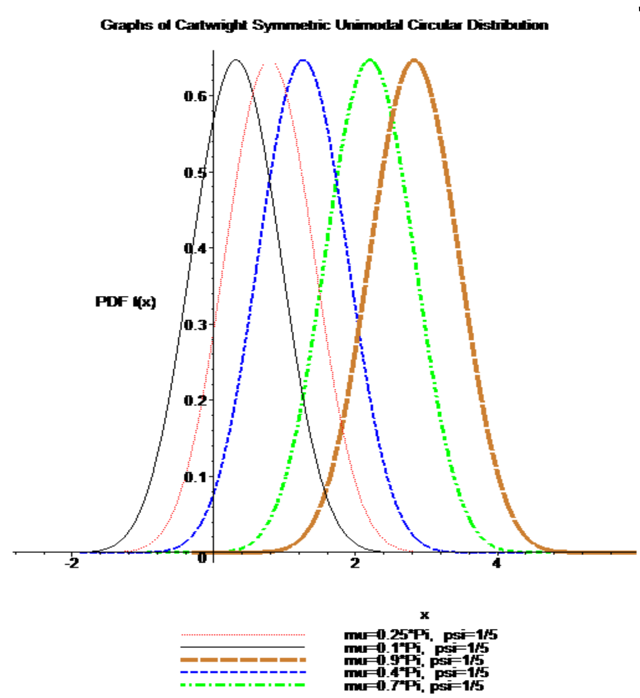

Figure 2. PDF $f(x)$ of $X \sim CPC(\mu, \psi)$, Eq. (1.3), when $\mu = 0.1\pi, 0.25\pi, 0.4\pi, 0.7\pi, 0.9\pi$, and $\psi = \frac{1}{5}$.

2. Distributional Properties

In this section, we will present some useful distributional properties of the Cartwright's distribution.

2.1. nth Moment

When n is a positive integer, using the pdf (1.3) of $X \sim CPC(\mu, \psi)$, the nth moment is given by

$$E(X^n) = \frac{2^{\left(\frac{2}{\psi}-1\right)} \Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi \Gamma\left(\frac{2}{\psi}+1\right)} \int_{\mu-\pi}^{\mu+\pi} x^n \left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}} dx, \quad \mu - \pi < x \le \mu + \pi. \tag{2.1}$$

Obviously, the integral on the right side of (2.1) cannot be evaluated analytically in closed form, and so should be solved numerically by some appropriate numerical quadrature rules as discussed before. However, using characteristic function of the Cartwright's distribution, we can easily find all of its moments, as described below.

2.2. Characteristic Function

The characteristic function $\Phi_X(t)$ of Cartwright's distribution is given by

$$\Phi_X(t) = E\left(e^{itX}\right), \quad -\infty < t < \infty, \quad i = \sqrt{-1},$$

$$= \frac{2^{\left(\frac{2}{\psi}-1\right)} \Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi \Gamma\left(\frac{2}{\psi}+1\right)} \int_{\mu-\pi}^{\mu+\pi} e^{itx} \cos^{\frac{2}{\psi}}\left(\frac{x-\mu}{2}\right) dx$$

$$= \frac{2^{\left(\frac{2}{\psi}\right)} \Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi \Gamma\left(\frac{2}{\psi}+1\right)} e^{it\mu} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i2tz} \cos^{\frac{2}{\psi}}(z)\, dz, \quad \left(\text{substituting } \frac{x-\mu}{2} = z\right)$$

from which using the Equation 3.892.2, page 476, of Gradshteyn and Ryzhik [12], and after simplification, we easily obtain the following characteristic function:

$$\Phi_X(t) = \begin{cases} 1, & t = 0 \\ \dfrac{\Gamma^2\left(\frac{1}{\psi}+1\right) e^{i\mu t}}{4\Gamma\left(\frac{1}{\psi}+1+t\right)\Gamma\left(\frac{1}{\psi}+1-t\right)}, & t \neq 0 \end{cases}, \tag{2.2}$$

where $\Gamma(.)$ denotes the gamma function. Now, since it's well-known that

$$E(X^n) = (-i)^n \left.\frac{d^n \Phi_X(t)}{dt^n}\right|_{t=0}, \tag{2.3}$$

all of the moments of Cartwright's distribution can easily be obtained by the differentiation of (2.2). Thus, using (2.2) and (2.3), and noting that $\frac{d\,\Gamma(z)}{d\,z} = \Gamma(z)\Psi(z)$, where $\Psi(z)$ denotes the digamma or psi function, the first and second moments, and variance are easily given by

$$E(X) = \frac{1}{4}\mu ,$$

$$E(X^2) = \frac{1}{4}\mu^2 - \frac{1}{2}\Psi\left(1, \frac{1}{\psi} + 1\right) ,$$

and

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{3}{16}\mu^2 - \frac{1}{2}\Psi\left(1, \frac{1}{\psi} + 1\right) ,$$

where $\Psi(n, z)$ is the *nth* polygamma function, which is the *nth* derivative of the digamma function $\Psi(z)$, where n is a positive integer**.** Similarly, we can obtain other moments using (2.2) and (2.3).

2.3. Median

Since the graph of the pdf of the Cartwright's distribution, is symmetric about its mean, $E(X) = \frac{1}{4}\mu$, its median, MD, is also be given by $MD = \frac{1}{4}\mu$.

2.4. Mode

The mode of the Cartwright's distribution is the value of $x = x_m$ (say), for which its pdf $f_\psi(x)$ given in (1.3) is maximum. Thus, differentiating $f_\psi(x)$ with respect to x, we have

$$\frac{df(x)}{dx} = -\frac{2^{\left(\frac{2}{\psi}-1\right)}\Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi\psi\Gamma\left(\frac{2}{\psi}+1\right)}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\left(\frac{2}{\psi}-1\right)}\sin\left(\frac{x-\mu}{2}\right),$$

which, when equated to 0, and solving for x, easily gives its mode. It is easy to observe that $\frac{d^2 f(x)}{dx^2} < 0$, when $x = x_m = \mu$, and thus the mode of the Cartwright's distribution, is $x_m = \mu$, and the maximum value of the pdf (1.3) is given by

$$f_X(x_m) = f_X(\mu) = \frac{2^{\left(\frac{2}{\psi}-1\right)}\Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi\Gamma\left(\frac{2}{\psi}+1\right)}.$$

Obviously, the Cartwright's distribution is unimodal.

2.5. Survival and Hazard Rate Functions

The survival (or reliability) and the hazard (or failure) rate functions of the Cartwright's distribution, $X \sim CPC(\mu, \psi)$, are respectively given by

$$R(x) = 1 - F(x) = 1 - \frac{2^{\left(\frac{2}{\psi}-1\right)} \Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi \Gamma\left(\frac{2}{\psi}+1\right)} \int_{\mu-\pi}^{x} \left[\cos\left(\frac{t-\mu}{2}\right)\right]^{\frac{2}{\psi}} dt$$

, $\mu - \pi < x \le \mu + \pi$, and

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\frac{2^{\left(\frac{2}{\psi}-1\right)} \Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi \Gamma\left(\frac{2}{\psi}+1\right)} \left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}}{1 - \frac{2^{\left(\frac{2}{\psi}-1\right)} \Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi \Gamma\left(\frac{2}{\psi}+1\right)} \int_{\mu-\pi}^{x} \left[\cos\left(\frac{t-\mu}{2}\right)\right]^{\frac{2}{\psi}} dt}, \quad \mu - \pi < x \le \mu + \pi .$$

The above equations cannot be evaluated analytically in closed forms, and hence should be solved numerically by some appropriate numerical quadrature rules. However, to describe the shapes of the hazard (or failure) rate functions of the Cartwright's distribution, their plots, for two different sets of values of the parameters, are provided below in Figures 3–4, by using Maple software. The increasing behaviors of the hazard (or failure rate) function, $h(x)$, are evident from the Figures 3 - 4. Also, from these figures, we notice that the hazard rate is concave up, that is, bathtub shaped, for all values of the parameters.

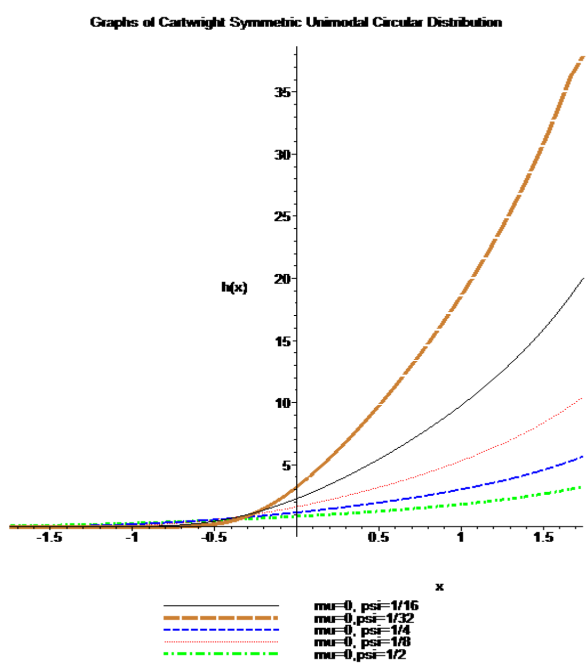

Figure 3. Hazard Function, $h(x)$ **, of** $X \sim CPC(\mu, \psi)$**, when** $\mu = 0$ **, and** $\psi = \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$.

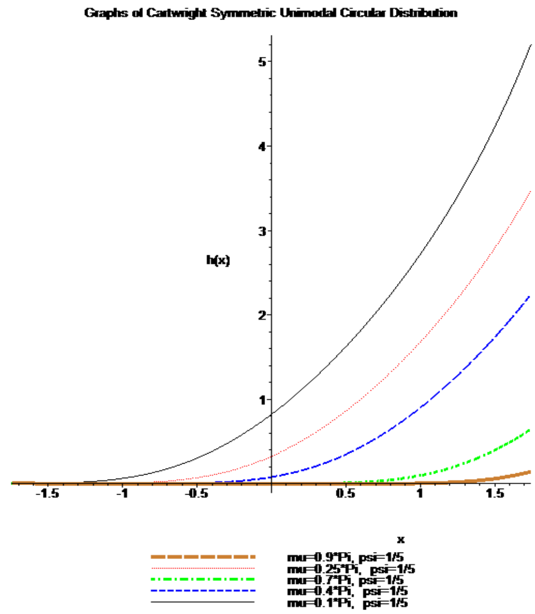

Figure 4. Hazard Function, $h(x)$**, of** $X \sim CPC(\mu,\psi)$**, when** $\mu = 0.1\pi, 0.25\pi, 0.4\pi, 0.7\pi, 0.9\pi$ **, and** $\psi = \frac{1}{5}$.

2.6. Entropy

It is well known that entropy offers an excellent tool to quantify the amount of information contained in a random observation regarding its parent distribution (population). A large value of entropy indicates the greater uncertainty in the data. As proposed by Shannon [20], entropy of an absolutely continuous random variable X having the probability density function $\phi_X(x)$ is defined as

$$H[X] = E[-\ln\{\phi_X(x)\}] = -\int_S \phi_X(x)\ln\{\phi_X(x)\}dx,$$

where $S = \{x \mid \varphi_X(x) > 0\}$. Using the above definition, the Shannon entropy of the Cartwright's distribution with the pdf (1.3), is given by

$$H[X] = E[-\ln\{f(x)\}]$$

$$= -\int_{\mu-\pi}^{\mu+\pi} C(\psi)\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}} \ln\left\{C(\psi)\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}\right\}dx, \qquad (2.4)$$

where $C(\psi)$ is the normalizing constant of $X \sim CPC(\mu,\psi)$, as computed in Section 1. Obviously, the integral I Eq. (2.4) cannot be evaluated analytically in closed form, and hence should be solved numerically by some appropriate numerical quadrature rules, such as the Newton-Cotes or Gaussian quadrature formulas. However, we have computed Shannon entropy, $H[X]$, of $X \sim CPC(\mu,\psi)$ for some selected values of the parameters by using Maple software, which are provided in Table 2.1 below.

Table 2.1 Shannon Entropy, $H[X]$, of $X \sim CPC(\mu, \psi)$

Parameters	**Shannon Entropy , $H[X]$**
$\mu = 0, \psi = 0.25$	1.010182
$\mu = 0, \psi = 0.5$	1.296303
$\mu = 0, \psi = 0.75$	1.441438
$\mu = 0, \psi = 1.00$	1.531024
$\mu = 0, \psi = 1.25$	1.591770
$\mu = 0, \psi = 1.50$	1.635400
$\mu = 0, \psi = 1.75$	1.668015

The effects of the parameters on Shannon entropy, $H[X]$, can easily be seen from the above Table 2.1. It is obvious from these computations that, when $\mu = 0$, Shannon entropy, $H[X]$, increases as ψ increases, that is, Shannon entropy, $H[X]$, is an increasing function of ψ, when $\mu = 0$. Similarly, we can compute the Shannon entropy, $H[X]$, for other values of the parameters, and study the effects of the parameters on Shannon entropy.

3. PERCENTILE POINTS

This section will provide the percentile points of the Cartwright distribution, with the pdf (1.3) and cdf (1.4). For any $0 < p < 1$, the $100\,p\,th$ percentile (also called the quantile of order p) of the Cartwright distribution, is a number x_p such that the area under $f_X(x)$ to the left of x_p is p. That is, x_p is any root of the equation given by $F(x_p) = \int_{\mu-\pi}^{x_p} f_X(u)\,du = p$. Using Maple software, we have computed the percentile points x_p for some selected values of p by taking two different sets of values of the parameters and have presented these in Table 3.1.

Table 3.1 Percentile Points of the Cartwright distribution, $X \sim CPC(\mu, \psi)$

Percentiles p / Parameters		**0.75**	**0.8**	**0.85**	**0.9**	**0.95**	**0.99**
$\mu = 0, \psi = 0.03125$	x_p	0.16787	0.20939	0.25774	0.31846	0.40817	0.57526
$\mu = 0, \psi = 0.0625$	x_p	0.23633	0.29470	0.36260	0.44767	0.57299	0.80473
$\mu = 0, \psi = 0.125$	x_p	0.33125	0.41279	0.50738	0.62555	0.79851	1.11370
$\mu = 0, \psi = 0.25$	x_p	0.46018	0.57275	0.70272	0.86394	1.09700	1.50947
$\mu = 0, \psi = 0.5$	x_p	0.62843	0.78030	0.95411	1.16680	1.46691	1.96856
$\mu = 0.1\pi, \psi = 0.2$	x_p	0.72869	0.83035	0.94794	1.09422	1.30672	1.68723

$\mu=0.25\pi,\psi=0.2$	x_p	1.19993	1.30160	1.41917	1.56545	1.77794	2.15841
$\mu=0.4\pi,\psi=0.2$	x_p	1.67040	1.77195	1.88938	2.03533	2.24693	2.62167
$\mu=0.7\pi,\psi=0.2$	x_p	2.50511	2.59544	2.69484	2.80870	2.94790	3.09534
$\mu=0.9\pi,\psi=0.2$	x_p	2.74229	2.81986	2.89731	2.97583	3.05673	3.12421

4. CHARACTERIZATIONS

Some characterizations of the proposed distributions are presented in this section. A significant number of researchers have examined the characterizations of both discrete and continuous probability distributions. To mention a few, Glänzel [10], Glänzel et al. [11], Kotz and Shanbhag [9], and Nagaraja [18], Galambos and Kotz [16], are notable. For more on the characterizations of probability distributions by the method of truncated moment, we refer our readers to Ahsanullah and Shakil [3], Ahsanullah et al. [5, 6]. These characterizations may assist as a foundation for parameter estimation and developing goodness-of-fit tests of distributions (Glänzel [10]). Since fitting of a particular probability distribution to the real-world data is an important area of research, it becomes necessary to justify whether the given probability distribution satisfies the underlying requirements by its characterizations. It appears from the literature that no attention has been paid to the characterizations of the Cartwright distribution. Therefore, this section presents some new characterizations of the Cartwright distribution. To verify our main results (Theorems 4.1 – 4.5), we will need some assumption and lemmas, which are provided in Appendix 1.

4.1. Characterizations by Truncated Moment

Here, we will provide the characterizations by the truncated moment method.

Theorem 4.1

Suppose that X is an absolutely continuous random variable having the cdf $F(x)$ with $F(\mu-\pi)=0$, and $F(\mu+\pi)=1$. Suppose that $E(X)$ exists. Suppose that the random variable X satisfies the Assumption 1 with $\gamma=\mu-\pi$ and $\delta=\mu+\pi$. Then, $E(X\mid X\le x)=g(x)\tau(x)$, where

$$g(x)=\frac{\int_{\mu-\pi}^{x}\theta f(\theta)d\theta}{f(x)}=\frac{\int_{\mu-\pi}^{x}\theta\left[\cos\left(\frac{\theta-\mu}{2}\right)\right]^{\frac{2}{\psi}}d\theta}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}},$$

and $\tau(x)=\dfrac{f(x)}{F(x)}$, if and only if

$$f_{\psi}(x)=\frac{2^{\left(\frac{2}{\psi}-1\right)}\Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi\Gamma\left(\frac{2}{\psi}+1\right)}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}},\quad \mu-\pi<x\le\mu+\pi,$$

which is the pdf of the Cartwright distribution.

Proof

Since $E(X|X\le x)=\dfrac{\int_{\gamma}^{x}\theta f(\theta)d\theta}{F(x)}$ and $\tau(x)=\dfrac{f(x)}{F(x)}$, we have $g(x)=\dfrac{\int_{\gamma}^{x}\theta f(\theta)d\theta}{f(x)}$.

Now, if the random variable X satisfies the Assumption 1 and has the Cartwright distribution, then we have

$$g(x)=\frac{\int_{\mu-\pi}^{x}\theta f(\theta)d\theta}{f(x)}=\frac{\int_{\mu-\pi}^{x}\theta\left[\cos\left(\frac{\theta-\mu}{2}\right)\right]^{\frac{2}{\psi}}d\theta}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}}.$$

Accordingly, the proof of "if" part of the Theorem 4.1 follows from Lemma 1. On the other hand, to prove the "only if" condition of Theorem 4.1, we suppose that

$$g(x)=\frac{\int_{\mu-\pi}^{x}\theta f(\theta)d\theta}{f(x)}=\frac{\int_{\mu-\pi}^{x}\theta\left[\cos\left(\frac{\theta-\mu}{2}\right)\right]^{\frac{2}{\psi}}d\theta}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}}.$$

Now, after differentiation of the above expression for $g(x)$ with respect to x and simplification, it follows that

$$g'(x)=x+\frac{\left(\int_{\mu-\pi}^{x}\theta\left[\cos\left(\frac{\theta-\mu}{2}\right)\right]^{\frac{2}{\psi}}d\theta\right)}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}}\frac{\frac{2}{\psi}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}-1}\sin\left(\frac{x-\mu}{2}\right)}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}}$$

$$=x+g(x)\frac{\frac{2}{\psi}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}-1}\sin\left(\frac{x-\mu}{2}\right)}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}}$$

or,

$$\frac{x-g'(x)}{g(x)}=-\frac{\frac{2}{\psi}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}-1}\sin\left(\frac{x-\mu}{2}\right)}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}}.$$

Now, following Lemma 1, we find

$$\frac{f'(x)}{f(x)}=\frac{x-g'(x)}{g(x)}=-\frac{\frac{2}{\psi}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}-1}\sin\left(\frac{x-\mu}{2}\right)}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}},$$

which, on integrating with respect to x, gives $f(x)=c\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}$, where c is a constant and need to be determined. Using the boundary conditions $F(\mu-\pi)=0$ and $F(\mu+\pi)=1$, and following the derivations as in Eq. (1.2), we have $c=\dfrac{2^{\left(\frac{1}{\psi}-1\right)}\Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi\Gamma\left(\frac{2}{\psi}+1\right)}$, $\psi>0$, and thus

$$f_\psi(x)=\frac{2^{\left(\frac{2}{\psi}-1\right)}\Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi\Gamma\left(\frac{2}{\psi}+1\right)}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}},\quad \mu-\pi<x\le\mu+\pi,$$

which is the pdf of the Cartwright distribution. This completes the proof of Theorem 4.1.

Theorem 4.2

If the random variable, X fulfils the Assumption 1 with $\gamma=\mu-\pi$ and $\delta=\mu+\pi$, then $E\left(X\,|X\ge x\right)=h(x)r(x)$, where $r(x)=\dfrac{f(x)}{1-F(x)}$, and

$$h(x)=\frac{\int_x^{\mu+\pi}\theta f(\theta)d\theta}{f(x)}=\frac{\int_x^{\mu+\pi}\theta\left[\cos\left(\frac{\theta-\mu}{2}\right)\right]^{\frac{2}{\psi}}d\theta}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}},$$

if and only if $f_\psi(x)=\dfrac{2^{\left(\frac{2}{\psi}-1\right)}\Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi\Gamma\left(\frac{2}{\psi}+1\right)}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}$, $\mu-\pi<x\le\mu+\pi$, which is the pdf of the Cartwright distribution.

Proof

The proof is similar to the Theorem 4.1, and easily follows from Lemma 2.

4.2. Characterizations by Order Statistics

Suppose $X_1, X_2, \ldots, X_n$ are n independent random variables with the continuous cumulative distribution function $F(x)$ and probability density function $f(x)$. We assume that $X_{1,n}\le X_{2,n}\le\ldots\le X_{n,n}$ are the corresponding order statistics. Then $X_{j,n}\,|\,X_{k,n}=x$, for $1\le k<j\le n$, is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V and has the probability density function, $f_V(v\,|\,x)$ where $f_V(v\,|\,x)=\dfrac{f(v)}{1-F(x)}$, $0\le v<x$, (see Arnold et al. [1], chapter 2). Further, $X_{i.,n}\,|\,X_{k,n}=x, 1\le i<k\le n$, is

distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where $f_W(w|x)=\dfrac{f(w)}{F(x)}$, $w<x$. Let $S_{k-1}=\dfrac{1}{k-1}\left(X_{1,n}+X_{2,n}+...+X_{k-1,n}\right)$, and $T_{k,n}=\dfrac{1}{n-k}\left(X_{k+1,n}+X_{k+2,n}+...+X_{n,n}\right)$.

Now, based on the order statistics, we will provide the characterizations of the Cartwright distribution in Theorems 4.3 and 4.4 below.

Theorem 4.3
Assume that the random variable X satisfies the Assumption 1 with $\gamma=\mu-\pi$ and $\delta=\mu+\pi$, then $E(S_{k-1}\,|\,X_{k,n}=x)=g(x)\tau(x)$, where $\tau(x)=\dfrac{f(x)}{F(x)}$ and

$$g(x)=\frac{\int_{\mu-\pi}^{x}\theta\left[\cos\left(\frac{\theta-\mu}{2}\right)\right]^{\frac{2}{\psi}}d\theta}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}},$$

iff $f_{\psi}(x)=\dfrac{2^{\left(\frac{2}{\psi}-1\right)}\Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi\Gamma\left(\frac{2}{\psi}+1\right)}\left[\cos\left(\dfrac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}$, $\mu-\pi<x\le\mu+\pi$, which is the probability density function of the Cartwright distribution.

Proof
It follows from David and Nagaraja [8], that

$$E(S_{k-1}\,|\,X_{k,n}=x)=E(X\,|\,X\le x).$$

Now, the result follows from Theorem 4.1.

Theorem 4.4
We assume that the random variable X satisfies the Assumption 1 with $\gamma=\mu-\pi$ and $\delta=\mu+\pi$, then $E(T_{k,n}\,|\,X_{k,n}=x)=h(x)r(x)$, where $r(x)=\dfrac{f(x)}{1-F(x)}$ and

$$h(x)=\frac{\int_{x}^{\mu+\pi}\theta f(\theta)d\theta}{f(x)}=\frac{\int_{x}^{\mu+\pi}\theta\left[\cos\left(\frac{\theta-\mu}{2}\right)\right]^{\frac{2}{\psi}}d\theta}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}},$$

if and only if $f_\psi(x) = \dfrac{2^{\left(\frac{2}{\psi}-1\right)}\Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi\Gamma\left(\frac{2}{\psi}+1\right)}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}$, $\mu-\pi < x \le \mu+\pi$, which is the probability density function of the Cartwright distribution.

Proof
It follows from David and Nagaraja [8], that

$$E(T_{k,n} \mid X_{k,n} = x) = E(X \mid X \ge x).$$

Now, the result follows from Theorem 4.2.

4.3. Characterization by Upper Record Values
Suppose that $X_1, X_2, \ldots$, is a sequence of independent and identically distributed absolutely continuous random variables with cumulative distribution function $F(x)$ and probability density function $f(x)$. Let $Y_n = \max(X_1, X_2, \ldots, X_n)$ for $n \ge 1$. We say that X_j is an upper record value of $\{X_n, n \ge 1\}$ if $Y_j > Y_{j-1}, j > 1$. The indices at which the upper records occur are given by the record times $\{U(n) > \min(j \mid j > U(n+1), X_j > X_{U(n-1)}, n > 1)\}$ and $U(1) = 1$. We will denote the *nth* upper record value as $X(n) = X_{U(n)}$. In the following theorem, we will provide the characterization of the Cartwright distribution based on upper record values.

Theorem 4.5
W assume that the random variable X satisfies the Assumption 1 with $\gamma = \mu - \pi$ and $\delta = \mu + \pi$, then $E(X(n+1) \mid X(n) = x) = h(x)r(x)$, where $r(x) = \dfrac{f(x)}{1-F(x)}$ and

$$h(x) = \frac{\int_x^{\mu+\pi}\theta f(\theta)d\theta}{f(x)} = \frac{\int_x^{\mu+\pi}\theta\left[\cos\left(\frac{\theta-\mu}{2}\right)\right]^{\frac{2}{\psi}}d\theta}{\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}},$$

if and only if $f_\psi(x) = \dfrac{2^{\left(\frac{2}{\psi}-1\right)}\Gamma^2\left(\frac{1}{\psi}+1\right)}{\pi\Gamma\left(\frac{2}{\psi}+1\right)}\left[\cos\left(\frac{x-\mu}{2}\right)\right]^{\frac{2}{\psi}}$, $\mu-\pi < x \le \mu+\pi$, which is the probability density function of the Cartwright distribution.

Proof
It follows from Nevzorov [19], that

$$E(X(n+1) \mid X(n) = x) = E(X \mid X \ge x).$$

Now, the result follows from Theorem 4.2.

5. Concluding Remarks

Cartwright [7] introduced a power-of-cosine circular distribution, which is used in the modelling of the directional spectra of ocean waves. Since fitting of a particular probability distribution with real life data is an important issue, it becomes necessary to justify whether the given probability distribution fulfils the essential requirements by its characterizations. Therefore, we have presented several distributional properties and some new characterizations of the Cartwright's power-of-cosine circular distribution in this paper. We sincerely believe that the findings of the paper will be beneficial to applied researchers in various fields.

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References

[1] Arnold, B.C., Balakrishnan, and Nagaraja, H. N. (2005). *First Course in Order Statistics*, Wiley, New York, USA.

[2] Ahsanullah, M. (2017). *Characterizations of Univariate Continuous Distributions*, Atlantis-Press, Paris, France.

[3] Ahsanullah, M., and Shakil, M. (2018). Some characterizations of raised cosine distribution. *International Journal of Advanced Statistics and Probability*, 6(2), 42 – 49.

[4] Ahsanullah, M., Kibria, B. M. G., and Shakil, M. (2014). *Normal and Student´s t Distributions and Their Applications*, Atlantis Press, Paris, France.

[5] Ahsanullah, M., Shakil, M., and Kibria, B. G. (2015). Characterizations of folded student's t distribution. *Journal of Statistical Distributions and Applications*, 2(1), 15.

[6] Ahsanullah, M., Shakil, M., and Kibria, B. M. (2016). Characterizations of Continuous Distributions by Truncated Moment. *Journal of Modern Applied Statistical Methods*, 15(1), 17.

[7] Cartwright, D. E. (1963), "The use of directional spectra in studying the output of a wave recorder on a moving ship, In *Ocean Wave Spectra*", pages 203 - 218, Prentice Hall, New Jersey, USA.

[8] David, H. A., and Nagaraja, H. N. (2003). *Order Statistics*, Third Edition, Wiley, New York, USA.

[9] Galambos, J., and Kotz, S. (1978). Characterizations of probability distributions. A unified approach with an emphasis on exponential and related models, *Lecture Notes in Mathematics*, 675, Springer, Berlin.

[10] Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. *Mathematical Statistics and Probability Theory* (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, 75 – 84.

[11] Glänzel, W., Telcs, A., and Schubert, A. (1984). Characterization by truncated moments and its application to Pearson-type distributions. *Z. Wahrsch. Verw. Gebiete*, 66, 173 – 183.

[12] Gradshteyn, I. S., and Ryzhik, I. M. (1990). *Table of integrals, series, and products*, Academic Press, Inc., San Diego, California, USA.

[13] Jammalamadaka, S. R., and Sengupta, A. (2001). *Topics in Circular Statistics*, World Scientific, Singapore.

[14] Jeffreys, H. (1961). *Theory of Probability*, 3rd ed., Oxford Classic Texts in the Physical Sciences, Oxford Univ. Press, Oxford, UK.

[15] Jones, M. C., and Pewsey, A. (2005). A family of symmetric distributions on the circle. *Journal of the American Statistical Association*, 100 (472), 1422 - 1428

[16] Kotz, S., and Shanbhag, D. N. (1980). Some new approaches to probability distributions. *Advances in Applied Probability*, 12, 903 - 921.

[17] Mardia, K. V., and Jupp, P. E. (2009). *Directional Statistics*, John Wiley & Sons, New York, USA.

[18] Nagaraja, H. (2006). Characterizations of Probability Distributions. In *Springer Handbook of Engineering Statistics* (pp. 79 - 95), Springer, London, UK.

[19] Nevzorov, V. B. (2001). *Records: Mathematical Theory*, Translation of Mathematical Monograph. American Mathematical Society, Rhode Island, USA.

APPENDIX 1

Assumption 1
Suppose the random variable X is absolutely continuous with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. We assume that $\gamma = \{x \mid F(x) > 0\}$ and $\delta = \inf\{x \mid F(x) < 1\}$. We further assume that $E(X)$ exists.

Lemma 1
Under the assumption 1, if $E(X \mid X \le x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_{\gamma}^{x} \frac{u - g'(u)}{g(u)} du$ is finite for all x, $\gamma < x < \delta$, then $f(x) = c e^{\int_{\gamma}^{x} \frac{u - g'(u)}{g(u)} du}$, where c is determined by the condition $\int_{\gamma}^{\delta} f(x) dx = 1$.

Proof
For proof, see Ahsanullah and Shakil [3].

Lemma 2
Under the assumption 1, if $E(X \mid X \ge x) = g(x) r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_{x}^{\delta} \frac{u + g'(u)}{g(u)} du$ is finite for all x, $\gamma < x < \delta$, then $f(x) = c e^{-\int_{\gamma}^{x} \frac{u + g'(u)}{g(u)} du}$, where c is determined by the condition $\int_{\gamma}^{\delta} f(x) dx = 1$.

Proof
For proof, see Ahsanullah and Shakil [3].