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Formulations and Benders Decomposition based Procedures for the Discrete Cost Multicommodity Network Design Problem

Imen MEJRI[1], Safa BHAR LAYEB[1] and Farah ZEGHAL MANSOUR[1]

[1]*Optimization and Analysis of Service and Industrial Systems Laboratory (OASIS)*
National Engineering School of Tunis, University of Tunis El Manar,Tunis, Tunisia

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Abstract: Multicommodity Network Design problems arise in a wide range of fields such as telecommunications, computer networks, supply chain and transportation. In this paper, we consider the Discrete Cost Multicommodity Network Design problem (DCMNDP) with several discrete facilities available for installation on each connection/edge. The DCMNDP requires the installation of at most one facility on each edge that allows routing the multicommodity flow demands in order to minimize the sum of the fixed facility installation costs. To solve the DCMNDP to optimality, we have tailored a Benders decomposition based procedure that we apply to two different formulations, namely the widely used arc-flow based formulation and the arc-path based formulation. This latter formulation is characterized by an exponential number of variables whose resolution requires the use of column generation as well as cut generation algorithms. An experimental computational study is conducted on real-world instances to compare the performance of the proposed formulations. The obtained results illustrate the effectiveness of applying Benders decomposition on the arc-path formulation.

Keywords: Benders decomposition; Column Generation; Network Design Problems; Multicommodity flows.

1. INTRODUCTION

We investigate the Discrete Cost Multicommodity Network Design problem (DCMNDP) defined as follows. Given a connected undirected graph $G = (V, E)$, where V is a set of n nodes and E is a set of m edges, let's consider a set of K multicommodity flow requirements defined between all node pairs, i.e. $K=n(n-1)/2$. For each commodity k, $k=1,..,K$, a preset flow demand d_k must be routed from a source node s_k to a sink node t_k; $s_k, t_k \in V$. Each commodity k, $k=1...K$, can be routed through several paths connecting s_k to t_k. For each edge e, $e \in E$, L_e denotes the number of physically distinct facilities that could be installed on edge e. In the field of telecommunications, facilities can represent different fiber optic cables with various bandwidth capacities. For each edge e, $e \in E$ and each facility l, $l=1,..., L_e$, let u^e_l denote the bidirectional capacity and let f^e_l denote the fixed design cost that corresponds to two convex and increasing step functions. The DCMNDP seeks a least-cost set of facilities to be installed on edges such that the prescribed K point-to-point commodity demand flows is routed simultaneously.

In the existing literature, Network Design Problems (NDPs) have received tremendous attention during the last decades. Namely, the design of minimum-cost multicommodity network with fixed and per unit costs, namely the multicommodity Capacitated Fixed-charge Network Design Problem (CFNDP), is a widely studied problem, e.g. [6, 9, 10, 13]. For this NDP variant, Rardin and Choe [21] have compared an arc-flow formulation and an arc-path formulation. They showed that no one formulation is better than the other. However, for the variant of network design problems without capacity (UCFND), they showed that the relaxation of the arc-flow formulation provides better lower bounds than those of the arc-path formulation. Recently, in 2018, Rahmaniani et al. [19] proposed several enhancement techniques to solve the arc-flow formulation for the stochastic multicommodity Capacitated Fixed-charge Network Design Problem (CFNDP). They reveal the performance of this formulation whenever used with an appropriate set of valid cuts.

However, to the best of our knowledge, rather limited works have considered a discrete set of bidirectional facilities to be installed on the edges while

E-mail: mejry.imen@gmail.com, safa.layeb@enit.utm.tn, farah.zeghal@enit.rnu.tn

involving fixed discrete costs. This latter variant of the Discrete Cost Multicommodity Network Design problem (DCMNDP) was first introduced by Minoux [14]. Gabrel et al. have developed some greedy heuristics [27] as well as exact methods [15, 26] to solve the arc-flow formulation. Moreover, exact well-designed Benders decomposition procedure based on the same formulation have been proposed by Mrad and Haouari [16]. Their proposed approach provided optimal solutions for instances with up to 50 nodes and 100 edges. A comprehensive survey of the applications of Benders decomposition to fixed charge network design problems was presented by Costa [3]. Later in 2017, Layeb et al. [22] have proposed and compared different compact MIP formulations for the DCMNDP and Ennaifer et al. [18] derived lower bounds by using a Lagrangian-based optimization approach. Recently, in 2018, Mejri et al. [11] have proposed a basic benders decomposition approach applied to an arc-flow formulation to solve optimally the DCMNDP. It is worthy referring the interested reader to the research of Balakrishnan et al. [1] for other related network design problems that are extensively investigated in the literature. For a survey on the efficiency of various methods used in literature to solve the different (NDPs) variants, we refer the reader to the recent paper of Wang [29].

In this paper, we develop an exact approach to solve efficiently the DCMNDP. Precisely, we propose a tailored Benders decomposition procedure that we applied to two different formulations: the well-known arc-flow formulation and an arc-path formulation. Because of the exponential number of its variables, we combine Benders decomposition with column generation approach to solve the proposed arc-path formulation. Valid inequalities are investigated to accelerate the convergence of the proposed Benders based approaches. An experimental study was conducted on randomly generated instances and real-world networks from the literature to compare the two formulations and evaluate the performance of the proposed approaches.

The paper remainder is stated as follows. Section 2 provides some real world applications of the DCMNDP. Section 3 describes the arc-flow formulation and details the proposed Benders decomposition procedure. Section 4 presents the arc-path formulation and its resolution approach based on Benders decomposition approach combined with column generation method. Section 5 investigates some valid inequalities proposed to accelerate the convergence of Benders decomposition procedure. Section 6 reports the computational results of the conducted experimental study. Finally, Section 7 presents the conclusions and suggestions for future studies.

2. Some Applications of the DCMNDP

The literature provides a large number of real life applications modeled as graph problems of designing capacitated networks. The DCMNDP has various real applications such as fiber-optic network design, train scheduling problem, and aircraft assignment. In this section, we briefly describe some DCMNDP's applications from telecommunication networks and aircraft assignment problems. The interested reader is referred to the paper of Minoux [15] for more detailed applications and to the recent applications survey paper on multicommodity network flow problem published by Wang in 2018 [28].

A. Phone Network Design

Nowadays, several companies, operating in the telecommunications field, are experiencing difficulties in improving the quality of their internal and external communications. Most companies aim to set up their own local network, and above all to guarantee to their employees, through this network, new services which offer more efficient communication. For this reason, phone operators should find the optimal investment policy for adding capacity to their networks [2]. The nodes of this network represent the customer nodes (distribution points in the telecommunication language) that are connected to the switching centers (also called central offices). The edges generally represent copper cables used to carry messages. The capacity u_e of the edge $e=\{i,j\} \in E$, represents the number of existing cables at the link that joins node i to node j; $i, j \in V$. For each edge $e \in E$ of the network, a fixed cost f_e is associated. This installation cost differs from one edge to another depending on the length, quality and location of the associated cable. Several types of facilities are available for each edge. Capacity can be established by installing different types of cables each with different capacities and costs. A flow value d_i is assigned for each node i, $i \in V$, this flow is measured by the number of cables required to connect this node to the switching center. For the multicommodity case, a set of K commodities are defined as a list of $k=1,..,K$, a flow of value d_k should be routed between the source node s_k (the distribution point) and the sink node t_k (the switching center). Thus, The DCMNDP requires installing the appropriate cables to route that traffic demands while minimizing the fixed installation costs.

B. Aircraft assignment problem

Let's consider the following aircraft assignment problem that appears in the air transport sector. In this setting, the nodes of the graph represent cities, and the connections between the nodes are defined by a number of arcs. Each arc represents a potential non-stop flight leg. Assume that the airline targets to satisfy K different origin-destination (or source-sink) demands corresponding to d_k passengers flying from an origin city s_k to a destination city t_k. The airline has L different aircraft types, i.e. $L_e=L$, $\forall e \in E$. Each aircraft l, $l=1,..,L$, is characterized by a capacity that represents an upper bound u_l on the maximal number of passengers it can carry, i.e. $u^e_l = u_l$, $\forall e \in E$. There is a fixed cost f^e_l of assigning fleet type l to leg e. The airline seeks a minimum cost of assignment of aircraft to legs.

3. Benders Decomposition Approach for the Basic Arc-Flow Formulation

A. Arc-Flow based Formulation

To formulate the DCMNDP, we begin by associating to each edge $e = \{i,j\} \in E$ two corresponding directed arcs (i,j) and (j,i). Let A be the derived set of arcs. Accordingly, we consider two types of decision variables. Let x_{ij}^k, $k=1,\dots,K$, $(i,j) \in A$, be a non-negative continuous variable representing the routed flow value of commodity k on arc (i,j). Then, let y^e_l, $l=1,..,L_e$, $e \in E$, be a binary variable taking the value 1 only if the facility l is installed on edge e. This yields to the following basic Arc-Flow formulation (**AF**) for the DCMNDP:

$$(AF): Minimize \sum_{e\in E}\sum_{l=1}^{L_e} f_l^e y_l^e \qquad (1)$$

Subject to:

$$\sum_{l=1}^{Le} y_l^e \le 1 \qquad \forall e \in E, \qquad (2)$$

$$\sum_{j:(i,j)\in A} x_{ij}{}^k - \sum_{j:(j,i)\in A} x_{ji}{}^k = \begin{cases} d_k \text{ if } i = s_k \\ 0 \text{ if } i \in V \setminus \{s_k, t_k\}, \forall k = 1,\dots,K, \\ -d_k \text{ if } i = t_k \end{cases} \qquad (3)$$

$$\sum_{k=1}^{K} x_{ij}{}^k + \sum_{k=1}^{K} x_{ji}{}^k \le \sum_{l=1}^{L_e} u_l^e y_l{}^e, \forall e = \{i,j\} \in E, \qquad (4)$$

$$x_{ij}{}^k \ge 0 \quad \forall (i,j) \in A, \quad k = 1,\dots,K, \qquad (5)$$

$$y_l{}^e \in \{0,1\}, \quad \forall e \in E, \quad l = 1,\dots,L_e. \qquad (6)$$

Objective (1) consists in minimizing the overall costs of installing facilities on the edges. Constraints (2) ensure that no more than one facility is installed on each edge. Equations (3) express the flow conservation constraints for each commodity k, $k=1,\dots,K$. Constraints (4) guarantee that the amount of flow circulating in both directions on each edge is not greater than the capacity of the facility installed on that edge. Constraints (5) and (6) describe the nature of variables x_{ij}^k and y^e_l.

It is worth noting here that for the NDPs, the linear relaxation of such Mixed Integer Linear Programming (MILP) formulation in general gives extremely weak lower bounds [6, 13, 25].

To find an optimal solution for the DCMNDP, we propose to solve the **AF** formulation using Benders decomposition approach.

B. Benders Decomposition Procedure for the AF Formulation

The Benders decomposition method [12] is a classic approach for NDPs. It consists in decomposing the problem into two problems called the master problem and the subproblem. Then, the resolution approach amounts to alternating between the Relaxed Master Problem (RMP) and the subproblem. Starting with the PMR, the algorithm successively solves the two problems and generates constraints amended to the PMR as it progresses through the resolution. The generated constraints are commonly called *Benders cuts*. This process is repeated until no constraints are generated and an optimal solution is found. For a review of the effectiveness of this method in solving NDPs, the interested reader is invited to consult [4, 5, 20].

The proposed (**AF**) formulation includes both integer variables associated to network design and continuous variables associated to routing demands. Therefore, this formulation can be decomposed into two problems using Benders decomposition: *(i)* a master problem containing the binary variables y and concerning the design of the network, and *(ii)* a subproblem involving continuous variables x for routing demands.

Let Γ be the set of solutions satisfying the network design constraints (2) and (6). For any $\tilde{y} \in \Gamma$, the following feasibility subproblem **SAF**($\tilde{y}$) involves only flow variables:

$$SAF(\tilde{y}): g(\tilde{y}) = Minimize \sum_{e\in E} \varepsilon_e \qquad (7)$$

Subject to: (3), (5),

$$\sum_{k=1}^{K} x_{ij}{}^k + \sum_{k=1}^{K} x_{ji}{}^k - \varepsilon_e \le \sum_{l=1}^{L_e} u_l{}^e \tilde{y}_l^e, \forall e = \{i,j\} \in E, \qquad (8)$$

$$\varepsilon_e \ge 0, \quad \forall e \in E, \qquad (9)$$

Where ε_e, $e \in E$, is a new variable associated to edge e. These artificial variables ensure the feasibility of the routing problem **SAF**($\tilde{y}$) [16].

Clearly, a solution $\tilde{y} \in \Gamma$ is feasible for model (**AF**) if and only if $g(\tilde{y})=0$, otherwise (i.e., $g(\tilde{y})>0$), solution $\tilde{y}$ violates a Benders cut that should be appended to the master program.

We consider α_{ik},, $i\in V$, $k =1,\dots,K$, and β_e, $e = \{i\ j\} \in E$, which represent the nonnegative dual variables related to Constraints (3) and (8), respectively. Therefore, by duality $g(\tilde{y})$ is given by :

$$g(\tilde{y}) = \sum_{k=1}^{K} d_k \alpha_{sk}^* - \sum_{e\in E} \beta_e^* \sum_{l=1}^{L_e} u_l^e \tilde{y}_l^e, \qquad (10)$$

where (α^*, β^*) represents the optimal dual solution of **SAF**$(\tilde{y})$.

Then, the basic benders decomposition procedure is as the following:

- **Step 1:** *Initialization*. Let $\mathbf{AF}_0$ be the model defined by (1), (2), and (6) and set $t = 0$.
- **Step 2**: *Master Problem Solution.* Solve $\mathbf{SAF}_t$ using a MILP solver. Let $\tilde{y}$ be an optimal solution to $\mathbf{AF}_t$.
- **Step 3**: *Subproblem Solution.* Solve **SAF**$(\tilde{y})$ using a MILP solver. Let (α^*, β^*) be an optimal dual solution of **SAF**$(\tilde{y})$.
- **Step 4**: *Optimality test.* If $g(\tilde{y})=0$, then $\tilde{y}$ is an optimal solution of **AF**. Otherwise, go to Step 5.
- **Step 5**: *Benders cut generation*. The following cut is generated:

$$\sum_{e\in E} \sum_{l=1}^{L_e} \beta_e^* u_l^e y_l^e \geq \sum_{k=1}^{K} d_k \alpha_k^* \qquad (11)$$

Let's define $\mathbf{AF}_{t+1}$ to be $\mathbf{AF}_t$ amended by the Benders cut (11).

Set $t \leftarrow t+1$ and go to Step 2.

This process is repeated until no violated Benders cut is generated and an optimal solution is found.

4. Benders Decomposition Approach for the Arc-Path Formulation

A. Arc-Path based Formulation

For each commodity k, $k=1...K$, we denote by P_k the set of all feasible paths between nodes s_k and t_k. Let a_{erk} be a binary constant that equals 1 if path $r\in P_k$ of commodity k includes edge e, and 0 otherwise. We define two types of decision variables: a continuous nonnegative variable z_r^k that represents the amount of flow circulating on the path $r\in P_k$ for each commodity k, $k=1,...,K$, from s_k to t_k, and a binary variable $y^e{}_l$ that models the decision to install a facility l on the edge e∈E, such that $y^e{}_l$ is equal to 1 if facility l is installed on the edge e, $l=1,...,L_e$, and 0 otherwise. Accordingly, a path-based formulation (**PF**) for the DCMNDP reads as:

$$(PF): Minimize \sum_{e\in E} \sum_{l=1}^{L_e} f_l^e y_l^e \qquad (12)$$

Subject to: (2), (6),

$$\sum_{r=1}^{|P^k|} z_r{}^k = d^k, \quad k=1,...,K, \qquad (13)$$

$$\sum_{k=1}^{K} \sum_{r=1}^{|P^k|} a_{er}{}^k z_r{}^k \leq \sum_{l=1}^{L_e} u_l{}^e y_l{}^e, \quad \forall e\in E, \qquad (14)$$

$$z_r{}^k \geq 0 \quad , \quad \forall r \in P^k, \quad k=1,...,K. \qquad (15)$$

Objective (12) is to minimize the total cost of installation. Constraints (13) ensure the routing of all flow demands from source nodes to sink nodes. Constraints (14) guarantee that the amount of flow circulating on each edge does not exceed the capacity of the facility installed on that edge. Constraints (15) are the non-negative constraints of variables z.

B. Benders Decomposition Procedure for the PF Formulation

In order to obtain optimal solutions for DCMNDP, we propose to solve the **PF** formulation using the Benders decomposition method. Actually, the **PF** formulation includes both integer variables associated with network design and continuous variables associated with the routing demands. Let's remind that Γ is the set of solutions satisfying the network design constraints (2) and (6). For any $\tilde{y} \in \Gamma$, the following feasibility subproblem **SPF**$(\tilde{y})$ involves only flow variables:

$$SPF(\tilde{y}): p(\tilde{y}) = Minimize \sum_{e\in E} \varepsilon_e \qquad (16)$$

Subject to: (13), (15),

$$\sum_{k=1}^{K} \sum_{r=1}^{|P_k|} a_{erk} z_r^k - \varepsilon_e \leq \sum_{l=1}^{L_e} u_l{}^e \tilde{y}_l^e, \forall e\in E, \qquad (17)$$

$$\varepsilon_e \geq 0, \quad \forall e \in E, \qquad (18)$$

Clearly, a solution $\tilde{y} \in \Gamma$ is feasible for formulation (**PF**) if and only if $p(\tilde{y})=0$, otherwise (i.e., $p(\tilde{y})>0$), solution $\tilde{y}$ violates a Benders cut that should be appended to the master program.

We consider $\varsigma=(\varsigma_k \geqslant 0, k=1,...,K)$ and $v=(v_e \geqslant 0, e\in E)$ which represent the nonnegative dual variables related to Constraints (13) and (17), respectively. Then, by duality $p(\tilde{y})$ is expressed as :

$$p(\tilde{y}) = \sum_{k=1}^{K} d_k \varsigma_k^* - \sum_{e\in E} v_e^* \sum_{l=1}^{L_e} u_l^e \tilde{y}_l^e, \qquad (19)$$

with (ς^*, v^*) represents the optimal dual solution of **SPF**$(\tilde{y})$.

Thus, the vector of network design variables $\tilde{y} \in \Gamma$ is feasible for the **PF** model if and only if Inequality (20) is verified:

$$\sum_{e \in E} v_e^* \sum_{l=1}^{L_e} u_l^e y_l^e \geq \sum_{k=1}^{K} d_k \varsigma_k^* \qquad (20)$$

If Constraint (20), representing the Benders cut, is violated, it should be added to the relaxed master problem.

In contrast to the **AF** formulation, the **PF** formulation cannot be solved directly using a MILP solver because of the exponential number of its variables *z*, thereby the use of column generation.

C. Column Generation for Generating Violated Benders Cuts

After solving the relaxed master problem and deriving a solution $\tilde{y} \in \Gamma$, it is necessary to verify if the selected capacities allow all the flow demands to be routed simultaneously. Therefore, we solve the subproblem **SPF**($\tilde{y}$) to demonstrate the optimality of the solution $\tilde{y}$ that was provided by the resolution of the relaxed master problem (i. e., $p(\tilde{y})=0$) or, if necessary, generate a violated Benders cut.

Using the column generation algorithm, for each commodity k, $k=1,\dots,K$, the reduced cost δ_{rk} associated with a path r, $r \in P_k$, can be formulated as :

$$\delta_{rk} = \sum_{e \in E} \alpha_{erk} \, v_e - \varsigma_k \quad \forall k = 1,\dots,K, \forall r \in P_k. \qquad (21)$$

The reduced cost of a path *r* Constraints (21) can be calculated by assigning to each arc (i,j) a cost $c_{ij}=v_{ij}=v_{ji}=v_e$, $e=\{i,j\} \in E$. Thus, the pricing subproblem amounts to a shortest path problem for each commodity k, $k=1,\dots,K$. These subproblems are solved by the Dijkstra algorithm, considering that all costs on the arcs are positive. Solving these problems generates a series of columns that are appended to the **SPF**($\tilde{y}$), if their reduced costs are negative. We reiterate with the column generation algorithm until there are no more columns with negative reduced cost.

The optimal primal and dual solutions (ε^*, z^*) and (ς^*, v^*) of the **SPF**($\tilde{y}$) are thus obtained and a violated Benders cut is identified if the value of the objective function is not zero (i. e., $p(\tilde{y})>0$).

Accordingly, a synthesis of the benders decomposition procedure is given below

- **Step 1:** *Initialization*. Let $\mathbf{PF}_0$ be the model defined by (1), (2), and (6) and set $t = 0$.
- **Step 2**: *Master Problem Solution.* Solve $\mathbf{PF}_t$ using a MILP solver. Let $\tilde{y}$ be an optimal solution to $\mathbf{PF}_t$.
- **Step 3**: *Subproblem Solution.* Solve **SPF**($\tilde{y}$) using a column generation algorithm. Let (ς^*, v^*) be an optimal dual solution of **SPF**($\tilde{y}$).
- **Step 4**: *Optimality test.* If $p(\tilde{y})=0$, then $\tilde{y}$ is an optimal solution of **PF**. Otherwise, go to Step 5.
- **Step 5**: *Benders cut generation.* A Benders cut according to Constraint (20) is generated.

  Let's define $\mathbf{PF}_{t+1}$ to be $\mathbf{PF}_t$ amended by the generated Benders cut.

  Set $t \leftarrow t+1$ and go to Step 2.

The first constraints denoted by (I1) are valid when all commodities between each pair of nodes; i.e. $K=n(n-1)/2$; should be routed. They express the connectivity of a graph implied by any feasible solution as described below :

$$\sum_{e \in E} \sum_{l=1}^{L_e} y_l^e \geq n-1 \qquad (22)$$

The second constraints denoted by (I2) require installing at least one facility on the edges that are incident to a commodity source or sink nodes. These constraints read as :

$$\sum_{e \in E: e=\{i,j\}} \sum_{l=1}^{L_e} y_l^e \geq 1, \quad \forall k = 1,\dots,K, \quad \forall i \in \{s_k, t_k\}. \qquad (23)$$

Then, the total capacity installed on the adjacent edges of each source or sink node should not be less than the prefixed flow demand, a third set of valid constraints (I3) can be written as follows:

$$\sum_{e=\{o_k,j\} \in E} \sum_{l=1}^{L_e} u_l^e y_l^e \geq d_k, \quad \forall k = 1,\dots,K, \quad \forall o_k \in \{s_k, t_k\}. \qquad (24)$$

The fourth constraints denoted by (I4) are cutest inequalities firstly introduced by Mrad and Haouari [15]. Precisely, let's consider a subset of nodes $R \subset V$ and $\bar{R}=V \backslash R$. We suppose that a cut $\delta(R)$ is represented by a set of edges, one extremity in R and the other extremity in $\bar{R}$. Assume that $d(R) = \sum_{k | R \cap \{s_k, t_k\} | = 1} d_k$ represents demands that must cross the cut δ(R). For each demand k, $k=1,\dots,K$, we start with a subset R including only the source node s_k. Then, we iteratively add the other adjacent nodes to the subset R until we find the sink node t_k. A valid cutset inequality is then derived as

$$\sum_{e \in \delta(R)} \sum_{l=1}^{L_e} u_l^e y_l^e \geq d(R) \ . \qquad (25)$$

5. COMPUTATIONAL RESULTS

To evaluate the performance of the proposed Benders decomposition approaches applied to the different formulations, we conducted a computational experimentation. Actually, the proposed exact approaches were implemented using C# language in concert with the MILP solver CPLEX 12.5. The computational experimentation was made on an i7 dual core 2.4 GHz Personal Computer with 12.0 GB RAM.

Computational tests were carried out on three instances test-bed sets. The first set of instances consists of 11 randomly generated instances (NET1-NET11) as described in [22] and the second set is composed of 6 real-life instances available in the NDPs literature [7, 17, 30]. It consists of 1 instance denoted by GRID12 and provided by France Telecom [17], 4 instances denoted by NSFNet (National Science Foundation Networks) [30] and 1 instance denoted by EON (European Optical Network) [7]. The physical topology of EON and NSFNet networks appear in Figures 1 and 2, respectively. The considered third test-bed consists of 7 instances derived from the Survivable Network Design Library (SNDlib) [24].

Fig. 1. EON network physical topology [7]

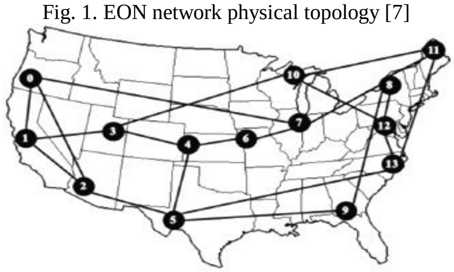

Figure. 2. NSFNet physical topology [30]

For the first and second set of instances, the numbers of nodes and edges range from 6 to 41, and 16 to 154, respectively, the number of facilities is equal to 2 and the number of commodities range from 8 to 132. For the third set of instances, the numbers of nodes and edges range from 11 to 17, and 21 to 42, respectively, the number of facilities range from 3 to 40 and the number of commodities range from 22 to 121. Table I displays the instances characteristics. For each instance, we report the instance designation (**Inst.**), the number of nodes (**n**), the number of edges (**m**), and the number of commodities (**K**).

TABLE I. THE INSTANCES CHARACTERISTICS

Inst.	***n***	***m***	***K***
NET1	6	16	8
NET2	6	16	8
NET3	6	16	8
NET4	6	16	8
NET5	8	26	13
NET6	8	26	13
NET7	9	26	12
NET8	9	26	12
NET9	41	154	77
NET10	41	154	77
NET11	41	154	77
GRID12	12	34	132
NSFNet1	14	42	21
NSFNet2	14	42	21
NSFNet3	14	42	21
NSFNet4	14	42	21
EON	19	72	36
Pdh_sndlib	11	34	24
Yuan	11	42	22
nobel_us_sndlib	14	21	91
nobel_germany_sndlib	17	26	121

We define the maximum CPU time limit at 3600 seconds and we execute the basic Benders decomposition procedure applied to each formulation. The objective of the first experimentations is to investigate the effectiveness of the proposed approach without considering the valid inequalities and to explore its performance when applied to the proposed formulations of the DCMNDP. Table II reports the obtained results. Let ***Sol.***, ***Time (s)***, and ***BCuts*** denote the value of the best solution, the total CPU time in seconds, and the number of generated Benders cuts, respectively. The last column (**Time ratio**) indicates the ratio of the CPU time of the **PF** formulation to the **AF** formulation.

TABLE II. PERFORMANCE OF THE BASIC BENDERS DECOMPOSITION PROCEDURE

Inst.	***AF** formulation*			***PF** formulation*			*Time ratio*
	Sol.	*Time (s)*	*BCuts*	*Sol.*	*Time (s)*	*BCuts*	
NET1	5	5.23	136	5	1.25	98	0.24
NET2	6	2.34	53	6	0.95	45	0.41
NET3	8	1.75	36	8	1.01	24	0.58
NET4	6	2.51	66	6	1.49	49	0.59
NET5	7	3471.60	4063	7	159.2	3698	0.05
NET6	7	71.47	676	7	13.9	348	0.19
NET7	8	549.23	743	8	99.36	478	0.18
NET8	7*	>3600	4064	8	2969.01	3147	-
NET9	12*	>3600	292	52	2178.01	187	-
NET10	24*	>3600	63	63	2836.1	42	-
NET11	14*	>3600	389	40	587.29	157	-
GRID12	23*	>3600	1108	24	4.97	924	-
NSFNet1	8*	>3600	1992	13	33.78	1285	-
NSFNet2	19*	>3600	566	21	387.58	395	-
NSFNet3	10*	>3600	1598	16	148.6	1098	-
NSFNet4	15*	>3600	955	19	98.4	397	-
EON	7*	>3600	1119	19	2963.1	746	-
Pdh_sndlib	9211368*	>3600	78	10023256*	>3600	59	-
Yuan	359359*	>3600	29	594900*	>3600	18	-
nobel_us_sndlib	153*	>3600	426	200*	>3600	348	-
nobel_germany_sndlib	189*	>3600	251	240*	>3600	147	-

*: Indicates that the optimum remained unfound after 3600 seconds

Table II shows that the basic Benders decomposition approach applied to the arc-flow formulation is unable to find optimal solutions for instances having more than 9 nodes and 26 edges. With the aforementioned procedure, only 7 small instances over the 21 instances are solved optimally within the 1 hour preset time. Concurrently, the basic Benders decomposition approach, in conjunction with the arc-path formulation, provides optimal solutions for 17 instances having up to 41 nodes and 154 edges within a total average CPU time of 734.35 seconds. Only 4 instances obtained from the Survivable Network Design Library [24] are still unsolved after one hour of CPU time computation. These results illustrate the potential of the **PF** formulation for solving small to medium scale instances.

In addition, we notice that the resolution of the **PF** formulation by the basic Benders decomposition approach requires less iterations (expressed by the **BCuts)** than those of the **AF** formulation.

To evaluate the impact of the proposed valid inequalities, we detail in Table III the solutions obtained by the enhanced Benders decomposition procedure in conjunction with both **AF** and **PF** formulations. For each instance, Table III reports the solution obtained by the enhanced Benders decomposition procedure (**Sol**), its gap to the optimal solution (**Gap(%)**), the total CPU time in seconds (**Time (s)**), and the number of generated Benders cuts (**BCuts**). The last two columns indicate the total number of Valid Inequalities (**VI**) and the ratio of **AF** CPU time in seconds over **PF** CPU time (**Time ratio**).

TABLE III. PERFORMANCE OF THE ENHANCED BENDERS DECOMPOSITION PROCEDURE

Inst.	***AF** formulation*				***PF** formulation*				*VI*	*Time ratio*
	Sol.	*Time(s)*	*BCuts*	*Gap(%)*	*Sol.*	*Time(s)*	*BCuts*	*Gap(%)*		
NET1	5	1.03	19	*0.00%*	5	0.5	0	*0.00%*	33	0.49
NET2	6	0.96	15	*0.00%*	6	0.2	8	*0.00%*	33	0.21
NET3	8	1.28	17	*0.00%*	8	0.34	14	*0.00%*	33	0.27
NET4	6	1.02	18	*0.00%*	6	0.15	12	*0.00%*	33	0.15
NET5	7	1506.8	18	*0.00%*	7	1.28	1	*0.00%*	53	0.00
NET6	7	43.39	13	*0.00%*	7	0.04	3	*0.00%*	53	0.00
NET7	8	202.40	1	*0.00%*	8	0.94	0	*0.00%*	49	0.00
NET8	8	0.40	1	*0.00%*	8	0.02	0	*0.00%*	49	0.05
NET9	35*	>3600	2097	32.69%	52	2698.3	1258	*0.00%*	309	-
NET10	42*	>3600	189	33.33%	63	1479.01	74	*0.00%*	309	-
NET11	38*	>3600	436	5.00%	40	1879.04	247	*0.00%*	309	-
GRID12	24	699.63	949	*0.00%*	24	236.7	597	*0.00%*	529	0.34
NSFNet1	13	6.97	111	*0.00%*	13	2.3	39	*0.00%*	85	0.33
NSFNet2	21	69.12	202	*0.00%*	21	18.96	73	*0.00%*	85	0.27
NSFNet3	16	951.15	1270	*0.00%*	16	285.01	982	*0.00%*	85	0.30
NSFNet4	19	167.21	391	*0.00%*	19	53.47	179	*0.00%*	85	0.32
EON	19	3604.87	2604	*0.00%*	19	1582	62	*0.00%*	145	0.44
Pdh_sndlib	10111368*	>3600	198	9.99%	11233089	159.3	0	*0.00%*	97	-
Yuan	498366*	>3600	169	24.10%	656600	298.4	0	*0.00%*	89	-
nobel_us_sndlib	208*	>3600	417	20.00%	260	287.17	0	*0.00%*	365	-
nobel_germany_sndlib	249*	>3600	598	22.19%	320	105.3	0	*0.00%*	485	-

As expected, the improved procedure exceeds the basic procedure in all cases. Table III shows that inequalities (I1), (I2), (I3) and (I4) clearly accelerate the Benders decomposition convergence for the arc-flow as well as the arc-path formulations. Interestingly, Table III illustrates that the enhanced Benders decomposition procedure when applied to **AF** formulation is able to find optimal solutions for 14 instances within an average CPU time of 518.30 seconds, and for all other instances, it provides approximate values which are less than 7% of the optimal solutions on average. Although, the **PF** formulation finds the optimal solutions for all tested instances within an average CPU time of 432.78 seconds. Indeed, the number of the derived Benders cuts has been reduced when the valid inequalities are applied. Moreover, for 5 instances, the enhanced Benders decomposition procedure considering the arc-path formulation ensures founding optimal solutions without generating any Benders Cut; i.e. **BCuts** *=0*.

6. CONCLUSION

The scope of this paper is to solve the challenging Discrete Cost Multicommodity Network Design problem (DCMNDP). Actually, such NP-hard problem is academically relevant as well as practically because of its real life applications in several fields such as telecommunications. To solve the DCMNDP to optimality, a Benders decomposition approach was developed and applied to two different formulations: the widely used arc-flow formulation and a proposed arc-path formulation. To accelerate the convergence of the tailored Benders based procedures, we investigate a set of valid inequalities that we appended to the appropriate relaxed master problem. The results of computational experiments, conducted on three test-beds of instances from the literature, illustrate the performance of the enhanced Benders decomposition procedure applied to the arc-path formulation. Actually, instances with up to 41 nodes and 154 edges were solved to optimality within reasonable CPU times. These promising results encourage investigating other valid inequalities and accelerating techniques for solving large-scale DCMNDP instances.

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Imen MEJRI received her PhD in Industrial Engineering from the OASIS-ENIT at the University Tunis el Manar (Tunisia). Her studies were related to network design and optimization problems. In 2012, she graduated from the National Engineering School of Bizerte, Tunisia, with a degree in Industrial Engineering. She is particularly interested in industrial engineering and artificial intelligence.

Safa BHAR LAYEB is a Polytechnic Engineer and has a Master thesis in Mathematical Engineering c. She is currently working as an Associate Professor of Industrial Engineering in The National Engineering School of Tunis, Tunisia. She is interested in industrial engineering, applied mathematics and operations research.

Farah ZEGHAL MANSOUR is currently a professor of Industrial Engineering in The National Engineering School of Tunis, Tunisia. She is particularly interested in operations research, Aircraft fleeting and routing, Healthcare optimization, and network design.