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Weisner's Method to Obtain Generating Functions for the Incomplete 2D Hermite Polynomials

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ABSTRACT

In this paper, we derive some generating functions for Incomplete 2D Hermite polynomials $h_{m,n}(x,y;\tau)$ by giving suitable interpretations to the indices (m) and (n) through Weisner's method.
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1.INTRODUCTION

The Incomplete 2D Hermite polynomials discussed in the present paper are characterized by two indices, two variables and one parameter. These polynomials are defined through the generating function (Dattoli 2003)

$$\exp(xu + yv + \tau uv) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{u^m}{m!}\frac{v^n}{n!}h_{m,n}(x,y;\tau), \tag{1.1}$$

where the polynomials $h_{m,n}(x,y,\tau)$ are explicitly provided by the series (Dattoli 2003)

$$h_{m,n}(x,y;\tau) = m!n!\sum_{r=0}^{\min(m,n)}\frac{\tau^r x^{m-r} y^{n-r}}{r!(m-n)!(n-r)!} \tag{1.2}$$

These polynomials satisfy the following simultaneous partial differential equations

$$\tau\frac{\partial^2 w}{\partial x^2} + y\frac{\partial w}{\partial x} - nw = 0 \tag{1.3}$$

and

$$\tau\frac{\partial^2 w}{\partial x \partial y} + x\frac{\partial w}{\partial x} - mw + w = 0 \tag{1.4}$$

Dattoli et al. 2000,2002 and 2003 introduced and discussed a theory of Incomplete 2D Hermite polynomials. Their link with Laguerre polynomials was discussed and it was shown that they are a useful tool to study quantum mechanical harmonic oscillator entagled states .The possibility of

developing the theory of complete 2D Hermite polynomials from the point of view of the incomplete forms was analyzed too. The orthogonality properties of the associated harmonic-oscillator functions were also discussed.
Recently, (Khan et al. 2008) derived some implicit summations formulae for incomplete 2D Hermite polynomials by using different analytical means on their respective generating functions [Dattoli, et al 2002; Dattoli, Ricce, 2003; Khan, et al 2008; Weisner, 1955].

In this paper, we have obtained new generating functions for the Incomplete 2D Hermite Polynomials by constructing a Lie algebra with the help of (Weisner's 1955) method by giving suitable interpretations to the indices (m) and (n) of the polynomials under consideration. The principal interest in the given results lies in the fact that a number of special cases listed in section 3 would yield many new results of the theory of special functions.

2. GROUP-THEORETIC METHOD

Replacing n by $p\frac{\partial}{\partial p}$ and m by $s\frac{\partial}{\partial s}$ in (1.3) and (1.4) respectively we get

$$\tau\frac{\partial^2 u}{\partial x^2}+y\frac{\partial u}{\partial x}-p\frac{\partial u}{\partial p}=0 \qquad (2.1)$$

and

$$\tau\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial u}{\partial x}-s\frac{\partial u}{\partial s}+u=0 \qquad (2.2)$$

We see that $u(x,y,p,s;\tau)=h_{m,n}(x,y;\tau)p^n s^m$ is a solution of (2.1) and (2.2), since $h_{m,n}(x,y;\tau)$ is a solution of (1.3) and (1.4).

We first consider the following first order linear differential operators

$A_1 = p\frac{\partial}{\partial p}$

$A_2 = s\frac{\partial}{\partial s}$

$A_3 = p^{-1}\frac{\partial}{\partial y}$

$A_4 = p\left(y+\tau\frac{\partial}{\partial x}\right)$

$A_5 = s^{-1}\dfrac{\partial}{\partial x}$

$A_6 = s\left(x + \tau\dfrac{\partial}{\partial y}\right)$

such that

$$A_1[h_{m,n}(x,y;\tau)p^n s^m] = nh_{m,n}(x,y;\tau)p^n s^m$$
$$A_2[h_{m,n}(x,y;\tau)p^n s^m] = mh_{m,n}(x,y;\tau)p^n s^m$$
$$A_3[h_{m,n}(x,y;\tau)p^n s^m] = nh_{m,n-1}(x,y;\tau)p^{n-1} s^m$$
$$A_4[h_{m,n}(x,y;\tau)p^n s^m] = h_{m,n+1}(x,y;\tau)p^{n+1} s^m$$
$$A_5[h_{m,n}(x,y;\tau)p^n s^m] = mh_{m-1,n}(x,y;\tau)p^n s^{m-1}$$
$$A_6[h_{m,n}(x,y;\tau)p^n s^m] = h_{m+1,n}(x,y;\tau)p^n s^{m+1}$$

where the operators $\{A_i : i = 1,2,3,4,5,6\}$ satisfy the following commutation relations

$[A_1,A_2]=0$ $[A_2,A_3]=0$ $[A_3,A_4]=1$
$[A_1,A_3]=-A_3$ $[A_2,A_4]=0$ $[A_3,A_5]=0$
$[A_1,A_4]=A_4$ $[A_2,A_5]=-A_5$ $[A_3,A_6]=0$
$[A_1,A_5]=0$ $[A_2,A_6]=A_6$
$[A_1,A_6]=0$
$[A_4,A_5]=0$ $[A_4,A_6]=0$ and $[A_5,A_6]=1$,

where [A,B]=AB-BA.

The above commutation relations show that the set of operators $\{A_i : i = 1,2,3,4,5,6\}$ generate a Lie-algebra λ and the sets of operators $\{A_1, A_3, A_4\}$ and $\{A_2, A_5, A_6\}$ form a sub algebras of λ.

It is clear that the differential operators

$$L_1 = \tau\frac{\partial^2}{\partial x^2} + y\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}$$

and

$$L_2 = \tau\frac{\partial^2}{\partial x \partial y} + x\frac{\partial}{\partial x} - s\frac{\partial}{\partial s} + 1$$

which can be expressed as:

$L_1 = A_3A_4 - n$ and $L_2 = A_5A_6 - m$

commutes with $\{A_i : i = 1,2,3,4,5,6\}$ that is

$$\begin{cases} [L_1, A_i] = 0 \quad , i = 1,2,3,4,5,6 \\ \text{and} \\ [L_2, A_i] = 0 \quad , i = 1,2,3,4,5,6 \end{cases} \tag{2.3}$$

The extended form of the groups generated by $\{A_i : i = 1,2,3,4,5,6\}$ are as follows:

$$e^{a_1 A_1} f(x, y, p, s; \tau) = f(x, y, pe^{a_1}, s; \tau),$$

$$e^{a_2 A_2} f(x, y, p, s; \tau) = f(x, y, p, se^{a_2}; \tau),$$

$$e^{a_3 A_3} f(x, y, p, s; \tau) = f(x, y + \frac{a_3}{p}, p, s; \tau)$$

$$e^{a_4 A_4} f(x, y, p, s; \tau) = \exp(a_4 py) f(x + a_4 p\tau, y, p, s; \tau),$$

$$e^{a_5 A_5} f(x, y, p, s; \tau) = f(x + \frac{a_5}{s}, y, p, s; \tau),$$

$$e^{a_6 A_6} f(x, y, p, s; \tau) = \exp(a_6 sx) f(x, y + a_6 s\tau, p, s; \tau)$$

where $f(x, y, p, s; \tau)$ is an arbitrary function.

Then we have

$$\begin{aligned} & e^{a_6 A_6} e^{a_5 A_5} e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1} f(x, y, p, s; \tau) = \\ & \exp(a_4 py + a_6 sx). f(x + a_4 p\tau + \frac{a_5}{s}, y + a_6 s\tau + \frac{a_3}{p}, pe^{a_1}, se^{a_2}; \tau) \end{aligned} \tag{2.4}$$

3. GENERATING FUNCTIONS

From the above discussion, we see that $u(x, y, p, s; \tau) = h_{m,n}(x, y; \tau) p^n s^m$ is a solution of the following systems

$$\begin{cases} L_1 u = 0 \\ (A_3 A_4 - n) u = 0 \end{cases} \qquad \begin{cases} L_2 u = 0 \\ (A_5 A_6 - m) u = 0 \end{cases}$$

From (2.3) , we easily see that
$SL_1(h_{m,n}(x, y; \tau) p^n s^m) = L_1 S(h_{m,n}(x, y; \tau) p^n s^m)$

and

$SL_2(h_{m,n}(x,y;\tau)p^n s^m) = L_2 S(h_{m,n}(x,y;\tau)p^n s^m)$, where

$$\mathrm{S} = e^{a_6 A_6} e^{a_5 A_5} e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1}$$

Therefore, the transformation $S(h_{m,n}(x,y;\tau)p^n s^m)$

is also annulled by L_1 and L_2.

By setting $\{a_i = 0 : i = 1,2,3,4; a_5 = a, a_6 = b\}$ and writing $f(x,y,p,s;\tau) = h_{m,n}(x,y;\tau)p^n s^m$ in (2.4), we get

$$e^{bA_6} e^{aA_5}(h_{m,n}(x,y;\tau)p^n s^m) = \exp(bsx)h_{m,n}(x+\frac{a}{s}, y+bs\tau;\tau)p^n s^m \tag{3.1}$$

but

$$e^{bA_6} e^{aA_5}(h_{m,n}(x,y;\tau)p^n s^m) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{b^l}{l!}\frac{a^k}{k!}(m(m-1)(m-2)...(m-k+1))h_{m-k+l,n}(x,y;\tau)p^n s^{m-k+l} \tag{3.2}$$

combining the above two relations (3.1) and (3.2), we get

$$\exp(bsx)h_{m,n}(x+\frac{a}{s}, y+bs\tau;\tau) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{b^l}{l!}\frac{a^k}{k!}(m(m-1)(m-2)...(m-k+1))h_{m-k+l,n}(x,y;\tau)s^{l-k}$$

where $|b| < \infty$ and $|a| < \infty$. (3.3)

If we put $a = 0, s = 1$ in equation (3.3), we get

$$\exp(bx)h_{m,n}(x, y+b\tau;\tau) = \sum_{l=0}^{\infty}\frac{b^l}{l!}h_{m+l,n}(x,y;\tau) \quad \text{where } |b| < \infty. \tag{3.4}$$

If we put $b = 0, s = 1$ in equation (3.3), we get

$$h_{m,n}(x+a, y;\tau) = \sum_{k=0}^{\infty}\frac{a^k}{k!}(m(m-1)(m-2)...(m-k+1))h_{m-k,n}(x,y;\tau) \quad \text{where } |a| < \infty. \tag{3.5}$$

Again by setting $\{a_i = 0 : i = 1,2,5,6; a_3 = c, a_4 = d\}$ and writing $f(x,y,p,s;\tau) = h_{m,n}(x,y;\tau)p^n s^m$ in (2.4) we get

$$e^{dA_4}e^{cA_3}(h_{m,n}(x,y;\tau)p^n s^m) = \exp(dpy)h_{m,n}(x+dp\tau, y+\frac{c}{p};\tau)p^n s^m \tag{3.6}$$

but

$$e^{dA_4}e^{cA_3}(h_{m,n}(x,y;\tau)p^n s^m) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{c^l}{l!}\frac{d^k}{k!}((n+k)(n+k-1)...(n+k-l+1))h_{m,\,n+k-l}(x,y;\tau)p^{n+k-l}s^m$$
(3.7)

combining the above two relations (3.6) and (3.7), we get

$$\exp(dpy)h_{m,n}(x+dp\tau, y+\frac{c}{p};\tau) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{c^l}{l!}\frac{d^k}{k!}((n+k)(n+k-1)...(n+k-l+1))h_{m,\,n+k-l}(x,y;\tau)p^{k-l}$$
where $|c| < \infty, |d| < \infty$. (3.8)

If we put $c = 0, p = 1$ in equation (3.8), we get

$$\exp(dy)h_{m,n}(x+d\tau, y;\tau) = \sum_{k=0}^{\infty}\frac{d^k}{k!}h_{m,\,n+k}(x,y;\tau) \text{ , where } |d| < \infty. \tag{3.9}$$

If we put $d = 0, p = 1$ in equation (3.8), we get

$$h_{m,n}(x, y+c;\tau) = \sum_{l=0}^{\infty}\frac{c^l}{l!}(n(n-1)...(n-l+1))h_{m,\,n-l}(x,y;\tau)\text{, where } |c| < \infty. \tag{3.10}$$

4. CONCLUSION:

We have seen that Weisner's group theoretic method is a power full tool in getting generating functions . It is also interesting to define a new function which forms generalization for the Incomplete 2D Hermite Polynomials under consideration and then by using Lie theoretic technique , we can obtain generating functions. We will deal with this aspect in the subsequent communication.

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استخدام طريقة ويزنر لايجاد الدوال المولدة لكثير الحدود هيرمايت ثنائي الابعاد الغير المكتملة

مهند جمال سعيد شهوان
قسم الرياضيات ، كلية العلوم ، جامعة البحرين

الملخص

فى هذا البحث نستمد بعض الدوال المولدة متعددة الحدود الهرمنية ذات البعدين غير المكتملة وذلك بإعطاء تفسيرات مناسبه للمؤشريين(m) و (n) من خلا طريقة فايزنز.