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ORIGINAL ARTICLE

Solving the class equation $x^d = \beta$ in an alternating group for each $\beta \in H \cap C^\alpha$ and $n \notin \theta$

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Abstract In this paper we find the solutions to the class equation $x^d = \beta$ in the alternating group A_n (i.e. find the solutions set $X = \{x \in A_n | x^d \in A(\beta)\}$) and find the number of these solutions $|X|$ for each $\beta \in H \cap C^\alpha$ and $n \notin \theta$, where $H = \{C^\alpha$ of $S_n | n > 1$, with all parts α_k of α different and odd$\}$. C^α is conjugacy class of S_n and form class C^α depends on the cycle partition α of its elements. If ($14 > n \notin \theta \cup \{9, 11, 13\}$) and $\beta \in H \cap C^\alpha$ in A_n, then F_n contains C^α, where $F_n = \{C^\alpha$ of $S_n |$ the number of parts α_k of α with the property $\alpha_k \equiv 3 \pmod 4$ is odd$\}$. In this work we introduce several theorems to solve the class equation $x^d = \beta$ in the alternating group A_n where $\beta \in H \cap C^\alpha$ and $n \notin \theta$ and we find the number of the solutions for n to be: (i) $14 > n \notin \theta$, (ii) $14 > n \notin \theta$ and $(n + 1) \notin \theta$, (iii) $14 > n \notin \theta$ and $C^\alpha \neq [1, 3, 7]$, (iv) $n = 9, 11, 13$, (v) $n > 14$.

1. Introduction

The main purpose of this work is to solve and find the number of solutions to the class equation $x^d = \beta$ in an alternating group, where β ranges over the conjugacy class $A(\beta)$ in A_n and d is a positive integer. In this paper we solve the class equation $x^d = \beta$ in A_n, where $\beta \in H \cap C^\alpha$ and $n \notin \theta = \{1, 2, 5, 6, 10, 14\}$ and we find the number of solutions when $H = \{C^\alpha$ of $S_n | n > 1$, with all parts α_k and α different and odd$\}$. C^α is conjugacy class of S_n. If $\lambda \in C^\alpha$ and $\lambda \notin H \cap C^\alpha$, then C^α does not split into the two classes $C^{\alpha\pm}$ of A_n. The Frobenius equation $x^d = c$ in finite groups was introduced by Frobenius (1903), and studied by many others, such as Ishihara et al. (2001), Takegahara (2002), Chigira (1996), who dealt with some types of finite groups, including finite cyclic groups, finite p-groups, and Wreath products of finite groups. Choose any $\beta \in S_n$ and write it as $\gamma_1\gamma_2 \ldots \gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ are the number of disjoint cycle factors including the 1-cycle of β. Since disjoint cycles commute, we can assume that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{c(\beta)}$ (Rotman, 1995). Therefore $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{c(\beta)})$ is a partition of n and it is call cycle type of β. Let $C^\alpha \subset S_n$ be the set of all elements with cycle type α, then we can determine the conjugate class of $\beta \in S_n$ by using cycle type of β, since each pair of λ and β in S_n are conjugate if they have the same cycle type (Zeindler, 2010). Therefore, the number of conjugacy classes of S_n is the number of partitions of n. However, this is not necessarily true in an alternating group. Let $\beta = (1\,2\,4)$ and $\lambda = (1\,4\,2)$ be two permutations in S_4 that belong to the same conjugacy class $C^\alpha = [1, 3]$ in S_4 (i.e. $C^\alpha(\beta) = C^\alpha(\lambda)$). Since $\alpha(\beta) = (\alpha_1(\beta), \alpha_2(\beta)) = (1, 3) = (\alpha_1(\lambda), \alpha_2(\lambda)) = \alpha(\lambda)$, they have the same cycle type, but λ and β are not conjugate in A_4. Let $\beta = (1\,2\,3)(4\,5\,6)(7\,8\,9)$ and $\lambda = (5\,3\,7)(1\,6\,9)(2\,4\,8)$ in S_9 where

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they belong to the same conjugacy class $C^\alpha = [3^3]$ in S_4 since $\alpha(\beta) = (3,3,3) = \alpha(\lambda)$. But here λ and β are conjugate in A_9. The first and second examples demonstrate that it is not necessary for two permutations to have the same cycle structure in order to be conjugate in A_n. In this work we discuss the conjugacy classes in an alternating group and we denote conjugacy class of β in A_n by $A(\beta)$.

Definition 1.1. A partition α is a sequence of nonnegative integers $(\alpha_1, \alpha_2, \ldots)$ with $\alpha_1 \geqslant \alpha_2 \geqslant \cdots$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. The length $l(\alpha)$ and the size $|\alpha|$ of α are defined as $l(\alpha) = Max\{i \in N; \alpha_i \neq 0\}$ and $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$. We set $\alpha \vdash n = \{\alpha \ partition; |\alpha| = n\}$ for $n \in N$. An element of $\alpha \vdash n$ is called a partition of n (Zeindler, 2010).

Remark 1.2. We only write the non zero components of a partition. Choose any $\beta \in S_n$ and write it as $\gamma_1\gamma_2 \ldots \gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ are the number of disjoint cycle factors including the 1-cycle of β. Since disjoint cycles commute, we can assume that $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_{c(\beta)}$. Therefore $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{c(\beta)})$ is a partition of n and each α_i is called part of α (see Zeindler, 2010).

Definition 1.3. We call the partition $\alpha = \alpha(\beta) = (\alpha_1(\beta), \alpha_2(\beta), \ldots, \alpha_{c(\beta)}(\beta))$ the cycle type of β (Zeindler, 2010).

Definition 1.4. Let α be a partition of n. We define $C^\alpha \subset S_n$ to be the set of all elements with cycle type α (Zeindler, 2010).

Definition 1.5. Let $\beta \in S_n$ be given. We define $c_m = c_m^{(n)} = c_m^{(n)}(\beta)$ to be the number of cycles of length m of β (Zeindler, 2010).

Lemma 1.6. *$C^{\alpha\pm}$ of A_n are ambivalent if and only if the number of parts α_k of α with the property $\alpha_k \equiv 3 \pmod 4$ is even (James et al., 1984).*

Remark 1.7

(1) The relationship between partitions and c_m is as follows: if $\beta \in C^\alpha$ is given then $c_m^{(n)}(\beta) = |\{i : \alpha_i = m\}|$ (see Zeindler (2010)).

(2) The cardinality of each C^α can be found as follows: $|C^\alpha| = \frac{n!}{z_\alpha}$ with $z_\alpha = \prod_{r=1}^{n} r^{c_r}(c_r)!$ and $c_r = c_r^{(n)}(\beta) = |\{i : \alpha_i = r\}|$ (see Bump (2004)).

(3) If x is a solution of $x^d = \beta$, d is a positive integer, and y is a conjugate to x, then y is a solution of $x^d = \lambda$, where λ is conjugate to β in an alternating group (or any finite group). We call $x^d = \beta$ a class equation in A_n, where β and x belong to conjugate classes in an alternating group (see Taban (2007)).

Definition 1.8. Let $\beta \in C^\alpha$, where β is a permutation in an alternating group. We define the $A(\beta)$ conjugacy class of β in A_n by:

$$A(\beta) = \{\gamma \in A_n \mid \gamma = t\beta t^{-1};\ \text{for some } t \in A_n\}$$
$$= \begin{cases} C^\alpha, & (\text{if } \beta \notin H) \\ C^{\alpha+} \text{ or } C^{\alpha-}, & (\text{if } \beta \in H) \end{cases}$$

where $H = \{C^\alpha \text{ of } S_n | \ n > 1$, with all parts α_k of α different and odd$\}$.

Remark 1.9

(1) $\beta \in C^\alpha \cap H^C \cap A_n \Rightarrow A(\beta) = C^\alpha$, where H^C is a complement set of H.

(2) $\beta \in C^\alpha \cap H \Rightarrow \beta \in A_n$ and C^α splits into the two classes $C^{\alpha\pm}$ of A_n.

(3) $\beta \in H \Rightarrow A(\beta) = \begin{cases} C^{\alpha+} & \text{if } \beta \in C^{\alpha+} \\ C^{\alpha-} & \text{if } \beta \in C^{\alpha-} \end{cases}$

(4) If $n \in \theta = \{1, 2, 5, 6, 10, 14\}$, then for each $\beta \in A_n$ we have β conjugate to its inverse in $A_n (\beta \underset{A_n}{\approx} \beta^{-1})$.

Definition 1.10. Let $F_n = \{C^\alpha \text{ of } S_n|$ the number of parts α_k of α with the property $\alpha_k \equiv 3 \pmod 4$ is odd$\}$. Then for each $\beta \in H \cap C^\alpha \cap F_n$ in S_n, we define $C^{\alpha\pm}$ of A_n by:

$$C^{\alpha+} = \{\lambda \in A_n \mid \lambda = \gamma\beta\gamma^{-1};\ \text{for some } \gamma \in A_n\} = A(\beta)$$
$$C^{\alpha-} = \{\lambda \in A_n \mid \lambda = \gamma\beta^{-1}\gamma^{-1};\ \text{for some } \gamma \in A_n\} = A(\beta^{-1})$$

Remark 1.11

(1) Suppose $n \notin \theta$ & $\beta \in H \cap C^\alpha$ in A_n, then we have:

(i) If $(n+1) \in \theta$, then $C^\alpha \neq [4]$ (since $H \cap [4] = \phi$).

(ii) If $(n+1) \in \theta$, and $C^\alpha \neq [n]$, then β does not conjugate to its inverse in A_n.

(iii) If $(n+1) \in \theta$ and $C^\alpha = [n]$, then $n = (9 \ or \ 13)$, and $(\beta \approx \beta^{-1})$. So we define $C^{\alpha\pm}$ by: $C^{\alpha+} = [n]^+ = A(\beta)$ and $C^{\alpha-} = \{[n] - A(\beta)\}$ or $\{A(\beta^\#)$; for some $\beta^\# \in [n]$ do not conjugate to $\beta\}$.

(2) Suppose $n \in \theta$ and $\beta \in H \cap C^\alpha$ in A_n, where $C^\alpha = [k_1, k_2, \ldots, k_L]$ and $k_i \neq 1$, $(1 \leqslant i \leqslant L)$ then

we have:

(i) $\beta \in H \cap [1, k_1, k_2, \ldots, k_L]$ in A_{n+1}.

(ii) $(\beta \underset{A_{n+1}}{\approx} \beta^{-1})$.

*Finally, based on (1) and (2), we consider for all $(14 > n \notin \theta)$ and $\beta \in H \cap C^\alpha$ in A_n, but $\beta \notin H \cap C^\alpha \cap F_n$ in $A_n \Rightarrow C^\alpha = [9]$ or $[13]$ or $[1,3,7]$. So we define $[1,3,7]^\mp$ by $[1,3,7]^+ = A(\beta)$ and $[1,3,7]^- = A(\beta^\#)$, where $\beta^\# \in [1,3,7]$ does not conjugate to β.

Theorem 1.12. *Let $A(\beta)$ be the conjugacy class of β in A_n, $\beta \in [K, L] \cap H$ and $14 > n \notin \theta$, where $[K, L]$ is a class of S_n. If p is a positive integer such that gcd $(p, K) = 1$ and gcd $(p, L) = 1$, then the solutions of $x^p \in A(\beta)$ in A_n are:*

(1) $[K, L]^-$ if $\beta^p = (\beta^{-1}$ or $\gamma)$, where γ is conjugate to β^{-1}.
(2) $[K, L]^+$ if $\beta^p = (\beta$ or $\gamma)$, where γ is conjugate to β.

Proof. Since $\beta \in A_n \cap [K, L] \cap H$, $[K, L]$ splits into two classes $[K, L]^\pm$ of A_n. However, $14 > n \notin \theta \Rightarrow [K, L] \in F_n \Rightarrow A(\beta) = [K, L]^+$ and $A(\beta^{-1}) = [K, L]^{-1}$. Also, since $\gcd(p, K) = 1$, p does not divide K, and since $\gcd(p, L) = 1$, p does not divide L. Then by Taban (2007, lemma 3.9) we have $[K, L]$ as the solution set of $x^p \in [K, L]$ in S_n.

(1) Assume $\beta^p = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [K,L]$. Then either $\lambda \in [K,L]^+$, or $\lambda \in [K,L]^-$.

If $\lambda \in [K,L]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$, $\lambda^p = t\beta^p t^{-1} = \left(\begin{array}{c} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{array}\right]$, $\lambda^p \in [K,L]^- \Rightarrow \lambda^p \notin [K,L]^+ = A(\beta)$.

If $\lambda \in [K,L]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$, $\lambda^p = t\beta^{-p}t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{array}\right]$, $\lambda^p \in [K,L]^+ = A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in is $[K,L]^-$.

(2) Assume $\beta^p = (\beta$ or $\gamma = b\beta b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [K,L]$. Then either $\lambda \in [K,L]^+$ or $\lambda \in [K,L]^-$.

If $\lambda \in [K,L]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$, $\lambda^p = t\beta^p t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{array}\right]$, $\lambda^p \in [K,L]^+ = A(\beta) \Rightarrow \lambda^p \notin [K,L]$.

If $\lambda \in [K,L]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$, $\lambda^p = t\beta^{-p}t^{-1} = \left(\begin{array}{c} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{array}\right]$, $\lambda^p \in [K,L]^- \Rightarrow \lambda^p \notin A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[K,L]^+$.

Lemma 1.13. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If p and q are two different prime numbers, $p | n$ and $q | n$, then there is no solution of $x^{pq} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^\pm$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.9) there is no solution of $x^{pq} \in [n]$ in S_n. So there is no solution of $x^{pq} \in A(\beta)$ in A_n.

Lemma 1.14. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If p and q are two different prime numbers, $p | n$ and q does not divide n, then there is no solution of $x^{pq} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^\pm$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.4) there is no solution of $x^{pq} \in [n]$ in S_n. So there is no solution of $x^{pq} \in A(\beta)$ in A_n.

Theorem 1.15. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If p and q are different prime numbers such that $gcd(p,n) = 1$ and $gcd(q,n) = 1$, then the solutions of $x^{pq} \in A(\beta)$ in A_n are:*

(1) $[n]^-$ if $\beta^{pq} = (\beta^{-1}$ or $\gamma)$, where γ is conjugate to β^{-1}.
(2) $[n]^+$ if $\beta^{pq} = (\beta$ or $\gamma)$, where γ is conjugate to β.

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^\pm$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$ and $A(\beta^{-1}) = [n]^-$. Also, since, $gcd(p,n) = 1$, p does not divide n, and $gcd(q,n) = 1$, q does not divide n. Then by Taban (2007, lemma 3.4) we have $[n]$ as the solution set of $x^{pq} \in [n]$ in S_n.

(1) Assume $\beta^{pq} = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$, $\lambda^{pq} = t\beta^{pq}t^{-1} = \left(\begin{array}{c} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{array}\right]$, $\lambda^{pq} \in [n]^- \Rightarrow \lambda^{pq} \notin [n]^+ = A(\beta)$.

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$, $\lambda^{pq} = t\beta^{-pq}t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \;] \\ tb\beta(tb)^{-1} \end{array}\right]$, $\lambda^{pq} \in [n]^+ = A(\beta)$. Then the solution set of $x^{pq} \in A(\beta)$ in A_n is $[n]^-$.

(2) Assume $\beta^{pq} = (\beta$ or $\gamma = b\beta b^{-1})$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$, $\lambda^{pq} = t\beta^{pq}t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{array}\right]$, $\lambda^{pq} \in [n]^+ = A(\beta) \Rightarrow \lambda^{pq} \notin [n]^-$.

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$, $\lambda^{pq} = t\beta^{-pq}t^{-1} = \left(\begin{array}{c} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{array}\right]$, $\lambda^{pq} \in [n]^- \Rightarrow \lambda^{pq} \notin A(\beta)$. Then the solution set of $x^{pq} \in A(\beta)$ in A_n is $[n]^+$.

Lemma 1.16. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If p is prime number such that $p | n$, then there is no solution of $x^p \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [K,L] \cap H$, $[K,L]$ splits into two classes $[K,L]^\pm$ of A_n. However, $14 > n \notin \theta \Rightarrow [K,L] \in F_n \Rightarrow A(\beta) = [K,L]^+$, then by Taban (2007, lemma 3.8) we have there is no solution of $x^p \in [K,L]$ in S_n. So there is no solution of $x^p \in A(\beta)$ in A_n.

Theorem 1.17. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$, and $\beta \in [K,L] \cap H$ where $[K,L]$ is a class of S_n. If $p | K$ and $p | L$, then there is no solution of $x^p \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [K,L] \cap H$, $[K,L]$ splits into two classes $[K,L]^\pm$ of A_n. However, $14 > n \notin \theta \Rightarrow [K,L] \in F_n \Rightarrow A(\beta) = [K,L]^+$, then by Taban (2007, lemma 3.8) we have there is no solution of $x^p \in [K,L]$ in S_n. So there is no solution of $x^p \in A(\beta)$ in A_n.

Theorem 1.18. *Let $A(\beta)$ be the conjugacy class of β in A_n, $\beta \in [n] \cap H$, $14 > n \notin \theta$ & $(n+1) \notin \theta$, where $[n]$ is a class of S_n. If p is a prime number such that $gcd(n,p) = 1$, then the solutions of $x^p \in A(\beta)$ in A_n are:*

(1) $[n]^-$ if $\beta^P = (\beta^{-1}$ or $\gamma)$, where γ is conjugate to β^{-1}.
(2) $[n]^+$ if $\beta^P = (\beta$ or $\gamma)$, where γ is conjugate to β.

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$ and $A(\beta^{-1}) = [n]^-$. Also, since $gcd(n,p) = 1$, p does not divide n. Then by Taban (2007, lemma 3.2) we have $[n] = [n]^+ \cup [n]^-$ as a solution of $x^p \in [n]$ in S_n. But, $[n] = A(\beta) \cup [n]^{-1}$ then the solution set of $x^p \in A(\beta)$ in A_n is either $[n]^-$ or $[n]^+$.

(1) Assume $\beta^p = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \left(\begin{array}{c} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{array} \right], \quad \lambda^p \in [n]^- \Rightarrow \quad \lambda^p \notin [n]^+ = A(\beta),$$

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{array} \right], \quad \lambda^p \in [n]^+ = A(\beta).$$

Then the solution set of $x^p \in A(\beta)$ in A_n is $[n]^-$.

(2) Assume $\beta^p = (\beta$ or $\gamma = b\beta b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{array} \right], \quad \lambda^p \in [n]^+ = A(\beta) \Rightarrow \quad \lambda^p \notin [n]^-,$$

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \left(\begin{array}{c} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{array} \right], \quad \lambda^p \in [n]^- \Rightarrow \quad \lambda^p \notin A(\beta).$$

Then the solution set of $x^p \in A(\beta)$ in A_n is $[n]^+$.

Example 1.19. Find the solutions of $x^5 \in A(1\ 3\ 2)$ in A_3.

Solution:

Since $(14 > 3 \notin \theta)$, $(1\ 3\ 2) \in [3] \cap H$, $gcd(3,5) = 1$, and $(1\ 3\ 2)^5 = (1\ 2\ 3) = (1\ 3\ 2)^{-1}$, then the solution of $x^5 \in A(1\ 3\ 2)$ in A_3 is $[3]^- = \{(1\ 2\ 3)\}$.

Lemma 1.20. *Let $A(\beta)$ be the conjugacy class of β in A_n, and $\beta \in [n] \cap H$, where $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $[n]$ is a class of S_n. If $p \mid n$, then there is no solution of $x^{pm} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.12) we have there is no solution of $x^{p^m} \in [n]$ in S_n. So there is no solution of $x^{p^m} \in A_{(\beta)}$ in A_n.

Lemma 1.21. *Let $A(\beta)$ be the conjugacy class of β in A_n, and $\beta \in [n] \cap H$, where $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $[n]$ is a class of S_n. If p and q are different prime numbers, $p \mid n$ and $q \mid n$ then there is no solution of $x^{p^m q^d} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.14) we have there is no solution of $x^{p^m q^d} \in [n]$ in S_n. So there is no solution of $x^{p^m q^d} \in A_\beta$ in A_n.

Lemma 1.22. *Let $A(\beta)$ be the conjugacy class of β in A_n, If p and q are two different prime numbers such that $p \mid n$, q does not divide n, and $\beta \in [n] \cap H$ where $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $[n]$ is a class of S_n, then there is no solution of $x^{p^m q^d} \in A(\beta)$ in A_n.*

Proof. Since$\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.15) we have there is no solution of $x^{p^m q^d} \in [n]$ in S_n. So there is no solution of $x^{p^m q^d} \in A(\beta)$ in A_n.

Lemma 1.23. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, where $[n]$ is a class of S_n. If p and q are different prime numbers such that $gcd(p,n) = 1$ and $gcd(q,n) = 1$, then the solutions of $x^{p^m q^d} \in A(\beta)$ in A_n are:*

(1) $[n]^-$ if $\beta^{P^m q^d} = (\beta^{-1}$ or $\gamma)$, where γ is conjugate to β^{-1}.
(2) $[n]^+$ if $\beta^{P^m q^d} = (\beta$ or $\gamma)$, where γ is conjugate to β.

Proof. $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, and since $gcd(p,n) = 1$, p does not divide n and $gcd(q,n) = 1 \Rightarrow q$ does not divide n. Then by Taban (2007, lemma 3.2) we have $[n] = [n]^+ \cup [n]^-$ as a solution set of $x^{p^m q^d} \in [n]$ in S_n. But, $[n] = A(\beta) \cup [n]^-$ then the solution set of $x^{p^m q^d} \in A(\beta)$ in A_n is either $[n]^-$ or $[n]^+$.

(1) Assume $\beta^{p^m q^d} = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^{p^m q^d} = t\beta^{(p^m q^d)}t^{-1} = \left(\begin{array}{c} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{array} \right],$$

$$\lambda^{p^m q^d} \in [n]^- \Rightarrow \lambda^{p^m q^d} \notin [n]^+ = A(\beta),$$

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^{p^m q^d} = t\beta^{-(p^m q^d)}t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{array} \right], \quad \lambda^{p^m q^d} \in [n]^+ = A(\beta).$$

Then the solution set of $x^{p^m q^d} \in A(\beta)$ in A_n is $[n]^-$.

(2) Assume $\beta^{p^m q^d} = (\beta$ or $\gamma = b\beta b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^{p^m q^d} = t\beta^{p^m q^d}t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{array} \right],$$

$$\lambda^{p^m q^d} \in [n]^+ = A(\beta) \Rightarrow \lambda^{p^m q^d} \notin [n]^-,$$

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^{p^m q^d} = t\beta^{-(p^m q^d)}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix},$$

$\lambda^{p^m q^d} \in [n]^- \Rightarrow \lambda^{p^m q^d} \notin A(\beta)$.

Then the solution set of $x^{p^m q^d} \in A(\beta)$ in A_n is $[n]^+$.

Lemma 1.24. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If $p_1, p_2, \ldots, p_m$ are different prime numbers such that, $p_i \mid n$, $\forall i = 1, \ldots, m$, then there is no solution for $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^\pm$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.17) we have there is no solution of $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in [n]$ in S_n. So there is no solution of $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in A(\beta)$ *in* A_n.

Lemma 1.25. *Let $A(\beta)$ be the conjugacy class of β in A_n, $\beta \in [n] \cap H$, and $14 > n \notin \theta$ & $(n+1) \notin \theta$, where is $[n]$ is a class of S_n. If $p_1, p_2, \ldots, p_m$ are different prime numbers and $p_i \mid n$ for some i, then there is no solution for $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in A(\beta)$* in A_n.

Proof. Since $\beta \in A_n \cap H$, $[n]$ splits into two classes of $[n]^\pm$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.19) we have there is no solution of $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in [n]$ in S_n. So there is no solution of $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in A(\beta)$ in A_n.

Lemma 1.26. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$, and $\beta \in [k_1, k_2, \ldots, k_l] \cap H$, where $[k_1, k_2, \ldots, k_l] \neq [1,3,7]$ is a class of S_n. If p is a prime number such that $gcd(p, k_i) = 1$, for each i, then the solutions of $x^p \in A(\beta)$ are:*

(1) $[k_1, k_2, \ldots, k_l]^-$ if $\beta^p = \beta^{-1}$ or γ, where γ is conjugate to β^{-1}.

(2) $[k_1, k_2, \ldots, k_l]^+$ if $\beta^p = \beta$ or γ, where γ is conjugate to β.

Proof. Since $\beta \in A_n \cap H \cap [k_1, k_2, \ldots, k_l]$, $[k_1, k_2, \ldots, k_l]$ splits into two classes $[k_1, k_2, \ldots, k_l]^\pm$ of A_n. However, $14 \geqslant n \notin \theta$ and $[k_1, k_2, \ldots, k_l] \neq [1,3,7] \Rightarrow [k_1, k_2, \ldots, k_l] \in F \Rightarrow A(\beta) = [k_1, k_2, \ldots, k_l]^+$, and $A(\beta^{-1}) = [k_1, k_2, \ldots, k_l]^-$. Also, since $gcd(p, k_i) = 1$ for each $\lambda[k_1, k_2, \ldots, k_l] \Rightarrow \lambda^p[k_1, k_2, \ldots, k_l]$. Then by Taban (2007, lemma 2.8) we have for each $\lambda \in [k_1, k_2, \ldots, k_l] \Rightarrow \lambda^p \in [k_1, k_2, \ldots, k_l]$.

(1) Assume $\beta^p = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [k_1, k_2, \ldots, k_l]$. Then either $\lambda \in [k_1, k_2, \ldots, k_l]^+$ or $\lambda \in [k_1, k_2, \ldots, k_l]^-$.

If $\lambda \in [k_1, k_2, \ldots, k_l]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix},$$

$\lambda^p \in [k_1, k_2, \ldots, k_1]^- \Rightarrow \lambda^p \notin [k_1, k_2, \ldots, k_1]^+ = A(\beta)$,

If $\lambda \in [k_1, k_2, \ldots, k_l]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix},$$

$\lambda^p \in [k_1, k_2, \ldots, k_1]^+ = A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[k_1, k_2, \ldots, k_1]^-$.

(2) Assume $\beta^p = (\beta$ *or* $\gamma = b\beta b^{-1}$ for some $b \in A_n)$, and let $\lambda \in [k_1, k_2, \ldots, k_1]$. Then either $\lambda \in [k_1, k_2, \ldots, k_1]^+$ or $\lambda \in [k_1, k_2, \ldots, k_1]^-$.

If $\lambda \in [k_1, k_2, \ldots, k_l]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix},$$

$\lambda^p \in [k_1, k_2, \ldots, k_1]^+ = A(\beta) \Rightarrow \lambda^p \notin [k_1, k_2, \ldots, k_l]^-$,

If $\lambda \in [k_1, k_2, \ldots, k_l]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix},$$

$\lambda^p \in [k_1, k_2, \ldots, k_1]^- \Rightarrow \lambda^p \notin A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[k_1, k_2, \ldots, k_1]^+$.

Remark 1.27. If there is no solution for $x^p \in [k_i]$, for some $1 \leqslant i \leqslant l$ then there exists no solution for $x^p \in [k_1, k_2, \ldots, k_l]$.

Lemma 1.28. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$, and $\beta \in [k_1, k_2, \ldots, k_l] \cap H$, where $[k_1, k_2, \ldots, k_l] \neq [1,3,7]$ is a class of S_n. If p is a prime number such that $p \mid k_i$, for some i, then no solution of $x^p \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap H \cap [k_1, k_2, \ldots, k_l]$, $[k_1, k_2, \ldots, k_l]$ splits into two classes $[k_1, k_2, \ldots, k_l]^\pm$ of A_n. Since $A(\beta) = [k_1, k_2, \ldots, k_l]^+$ and $p \mid k_i$, then by Taban (2007, lemma 3.1) we have no solution for $x^p \in [k_i]$ in S_n. Then no solution for $x^p \in [k_1, k_2, \ldots, k_l]$ in. So no solution for $x^p \in A(\beta)$ in A_n.

Lemma 1.29. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$, and $\beta \in [k_1, k_2, \ldots, k_l] \cap H$, where $[k_1, k_2, \ldots, k_l] \neq [1,3,7]$ is a class of S_n, P is a prime number, is a positive integer. If for some $1 \leqslant i \leqslant l$ such that $p \mid k_i$, then we have no solution of $x^{p^m} \in A\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap H \cap [k_1, k_2, \ldots, k_l]$, $[k_1, k_2, \ldots, k_l]$ splits into two classes $[k_1, k_2, \ldots, k_l]^\pm$ of A_n. Since $A(\beta) = [k_1, k_2, \ldots, k_l]^+$ and $p \mid k_i$, then by Taban (2007, lemma 3.12) we have no solution for $x^{p^m} \in [k_i]$ in S_n. So no solution for $x^{p^m} \in [k_1, k_2, \ldots, k_l]$ in S_n. Then no solution for $x^{p^m} \in A(\beta)$ in A_n.

Definition 1.30. Let $\beta \in [9]$ of S_9, where $\beta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$. We define class $[9]^+$ of A_9 by $A(\beta) = [9]^+ = \{\mu \in [9] \| \mu = t\beta t^{-1};$ for some $t \in A_9\}$.

Remark 1.31

(i) $[9]^{-1} = [9] - A(\beta) = \{\mu \in [9] \| \mu \neq t\beta\, t^{-1}; \textit{for all } t \in A_9\}$.

(ii) Let $\beta \in [9]$ of S_9 where $\beta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$, $\bar{\beta} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ $\bar{\bar{\beta}} = (a_1, a_5, a_9, a_4, a_8, a_3, a_7, a_2, a_6)$ and d is a positive integer we have:

(1)	$\beta^d = \beta \iff d \equiv 1 \pmod 9$
(2)	$\beta^d = \bar{\beta} \iff d \equiv 2 \pmod 9$
(3)	$\beta^d = \bar{\bar{\beta}} \iff d \equiv 4 \pmod 9$
(4)	$\beta^d = \bar{\bar{\beta}}^{-1} \iff d \equiv 5 \pmod 9$
(5)	$\beta^d = \bar{\beta}^{-1} \iff d \equiv 7 \pmod 9$
(6)	$\beta^d = \beta^{-1} \iff d \equiv 8 \pmod 9$

(iii) (1) $A(\beta) = A(\beta^{-1})$, $A(\bar{\beta}) = A(\bar{\bar{\beta}}^{-1})$, $A(\bar{\bar{\beta}}) = A(\overline{\overline{\beta^{-1}}})$ [Since for each $\lambda \in [9]^{\mp}$ in A_9, where $\lambda = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9)$, $\exists \mu = (b_1, b_8)\ (b_2, b_7)\ (b_3, b_6)\ (b_4, b_5) \in A_9$ such that $\mu\lambda\mu^{-1} = \lambda^{-1}$].

(2) $A(\beta) = A(\bar{\beta})$ since $\exists t = (a_3, a_6)\ (a_1, a_5, a_7, a_8, a_4, a_2) \in A_9$ such that $t\beta t^{-1} = \bar{\beta}$.

(3) $A(\beta^{-1}) = A(\bar{\bar{\beta}})$ since $\exists t = (a_3, a_9)\ (a_1, a_8, a_7, a_2, a_4, a_5) \in A_9$ such that $t\bar{\bar{\beta}}t^{-1} = \beta^{-1}$.

Theorem 1.32. *Let $\beta \in [9]$ of S_9. If d is a positive integer such that $gcd(d,9) = 1$, then the solutions of $x^d \in A(\beta)$ in A_9 are $A(\beta)$.*

Proof. Since $\beta \in [9] \cap H \cap A_9$, [9] splits into two classes $A(\beta)$ & $[9]^-$ of A_9 and $gcd(d,9) = 1$, then d does not divide 9. Then by Taban (2007, lemma 3.2), the solution set of $x^d \in [9]$ in S_9 is [9]. For each $\lambda \in [9]$ we have $\alpha \in A(\beta)$ *or* $\alpha \notin A(\beta)$. If, $\alpha \in A(\beta)$ we have $\lambda \underset{A_9}{\approx} \beta(\lambda$ conjugate to β in $A_9) \Rightarrow \lambda^d \underset{A_9}{\approx} \beta^d$. How-ever, $\beta^d \underset{A_9}{\approx} \beta \Rightarrow \lambda^d \underset{A_9}{\approx} \beta \Rightarrow \lambda^d \in A(\beta)$. If $\alpha \notin A(\beta)$, assume $\lambda^d \in A(\beta) \Rightarrow \lambda^d \underset{A_9}{\approx} \beta$. But $\beta \underset{A_9}{\approx} \beta^d \Rightarrow \lambda^d \underset{A_9}{\approx} \beta^d \Rightarrow \lambda \underset{A_9}{\approx} \beta \Rightarrow$, (which is a contradiction). Then the solution of in A_9 is $A(\beta)$.

Definition 1.33. Let $\beta = \gamma\lambda \in [1,3,7]$ of S_{11}, where $\gamma = (b_1, b_2, b_3)$, $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$. We define classes $[1,3,7]^{\pm}$ of A_{11} by:

$A(\beta) = [1,3,7]^+ = \{\mu \in [1,3,7] \| \mu = t\beta t^{-1};$ *for some* $t \in A_{11}\}$ and $A(\overset{\#}{\beta}) = [1,3,7]^- = \{\mu \in [1,3,7] \mid \mu = t\,\overset{\#}{\beta}\,t^{-1};$ *for some* $t \in A_{11}\}$ where $\overset{\#}{\beta} = \gamma\bar{\lambda}$ and $\bar{\lambda} = (a_1, a_4, a_7, a_3, a_6, a_2, a_5)$.

Remark 1.34

(i) Let $\beta = \gamma\lambda \in [1,3,7]$ of S_{11} where $\gamma = (b_1, b_2, b_3)$, $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$, $\bar{\lambda} = (a_1, a_4, a_7, a_3, a_6, a_2, a_5)$, $\bar{\bar{\lambda}} = (a_1, a_3, a_5, a_7, a_2, a_4, a_6)$, and d is a positive integer number. We have:

(1)	$\beta^d = \beta \iff d \equiv 1 \pmod{21}$
(2)	$\beta^d = \gamma^{-1}\bar{\bar{\lambda}} \iff d \equiv 2 \pmod{21}$
(3)	$\beta^d = \gamma\bar{\lambda}^{-1} \iff d \equiv 4 \pmod{21}$
(4)	$\beta^d = \gamma^{-1}\bar{\bar{\lambda}}^{-1} \iff d \equiv 5 \pmod{21}$
(5)	$\beta^d = \gamma^{-1}\lambda \iff d \equiv 8 \pmod{21}$
(6)	$\beta^d = \gamma\bar{\lambda} \iff d \equiv 10 \pmod{21}$
(7)	$\beta^d = \gamma^{-1}\bar{\lambda}^{-1} \iff d \equiv 11 \pmod{21}$
(8)	$\beta^d = \gamma\lambda^{-1} \iff d \equiv 13 \pmod{21}$
(9)	$\beta^d = \gamma\bar{\bar{\lambda}} \iff d \equiv 16 \pmod{21}$
(10)	$\beta^d = \gamma^{-1}\bar{\lambda} \iff d \equiv 17 \pmod{21}$
(11)	$\beta^d = \gamma\bar{\bar{\lambda}}^{-1} \iff d \equiv 19 \pmod{21}$
(12)	$\beta^d = \beta^{-1} \iff d \equiv 20 \pmod{21}$

(ii) (1) $A(\beta) = A(\beta^{-1})$, $A(\gamma^{-1}\bar{\bar{\lambda}}) = A(\gamma\bar{\bar{\lambda}}^{-1})$, $A(\gamma\bar{\lambda}^{-1}) = A(\gamma^{-1}\bar{\lambda})$, $A(\gamma^{-1}\bar{\bar{\lambda}}^{-1}) = A(\gamma\bar{\bar{\lambda}})$, $A(\gamma^{-1}\lambda) = A(\gamma\lambda^{-1})$, $A(\gamma\bar{\lambda}) = A(\gamma^{-1}\bar{\lambda}^{-1})$ [Since for each $\beta = \gamma\lambda \in [1,3,7]$ in A_{11} where $\gamma = (b_1, b_2, b_3)$, $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$, $\exists \mu = (b_1, b_3)\ (a_2, a_7)\ (a_3, a_6)\ (a_4, a_5) \in A_{11}$ such that $\mu\lambda\mu^{-1} = \lambda^{-1}$].

(2) $A(\beta) = A(\gamma^{-1}\bar{\lambda})$ [since $\exists t = (b_2, b_3)\ (a_1, a_4, a_5, a_3, a_7, a_6) \in A_{11}$ such that $t\beta t^{-1} = \gamma^{-1}\bar{\lambda}$].

(3) $A(\beta) = A(\gamma\bar{\bar{\lambda}})$ [since $\exists\ t = (a_1, a_3, a_4)\ (a_7, a_6, a_2) \in A_{11}$ such that $t\beta t^{-1} = \gamma\bar{\bar{\lambda}}$].

(4) $A(\gamma\bar{\bar{\lambda}}^{-1}) = A(\gamma\bar{\lambda})$ [since $\exists t = (a_1, a_4, a_2)\ (a_3, a_5, a_6) \in A_{11}$ such that $t\gamma\bar{\bar{\lambda}}^{-1}t^{-1} = \gamma\bar{\lambda}$].

(5) $A(\gamma\bar{\bar{\lambda}}^{-1}) = A(\gamma\lambda^{-1})$ [since $\exists t = (a_1, a_6, a_5)\ (a_2, a_3, a_7) \in A_{11}$ such that $t\gamma\lambda^{-1}t^{-1} = \gamma\bar{\bar{\lambda}}^{-1}$].

Theorem 1.35. *Let $L = \{m \in N \mid m \equiv q \pmod{21};$ for some $q = 1, 4, 5, 16, 17, 20\}$. If d is a positive integer such that $gcd(d,3) = 1$ & $gcd(d,7) = 1$ and $\beta \in [1,3,7]$ of S_{11}, then the solutions of $x^d \in A(\beta)$ in A_{11} are:*

(1) $A(\beta)$ if $d \in L$.

(2) $A(\overset{\#}{\beta})$ if $d \notin L$.

Proof. Since $\beta \in [1,3,7] \cap H \cap A_{11}$, $[1,3,7]$ splits into two classes $A(\beta)$ & $A(\overset{\#}{\beta})$ of A_{11}, $gcd(d,3) = 1$ and $gcd(d,7) = 1$, then d does not divide 3 and d does not divide 7. Then by Taban (2007, lemma 2.8) we have $[1,3,7] = A(\beta) \cup A(\overset{\#}{\beta})$ as a solution set of $x^d \in [1,3,7] = A(\beta) \cup A(\overset{\#}{\beta})$ in S_{11}. However, $A(\beta) \cap A(\overset{\#}{\beta}) = \phi$, so for each $\pi \in [1,3,7] \Rightarrow (\pi \in A(\beta)\ \&\ \pi \notin A(\overset{\#}{\beta}))$ or $(\pi \in A(\overset{\#}{\beta})\ \&\ \pi \notin A(\beta))$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then we have $(\pi \underset{A_{11}}{\approx} \beta)$ π conjugate to β in A_{11}. However, $\pi^d \underset{A_{11}}{\approx} \pi$ (since $d \in L) \Rightarrow \pi^d \underset{A_{11}}{\approx} \beta \Rightarrow \pi^d \in A(\beta)$ & $\pi^d \notin A(\overset{\#}{\beta})$. If $\pi \in A(\overset{\#}{\beta})$, we have $(\pi \underset{A_{11}}{\approx} \overset{\#}{\beta})$. But $\pi^d \underset{A_{11}}{\approx} \pi$ (since $d \in L) \Rightarrow \pi^d \underset{A_{11}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A(\overset{\#}{\beta})$ & $\pi^d \notin A(\beta)$. Then the solution set of $x^d \in A(\beta)$ in A_{11} is $A(\beta)$.

(2) Assume $d \notin L$. If $\pi \in A(\beta)$, then we have $(\pi \underset{A_{11}}{\approx} \beta) \Rightarrow \overset{\#}{\pi} \underset{A_{11}}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_{11}}{\approx} \overset{\#}{\pi}$ (since $d \notin L) \Rightarrow \pi^d \underset{A_{11}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A(\overset{\#}{\beta})$ & $\pi^d \notin A(\beta)$. If $\pi \in A(\overset{\#}{\beta}) \Rightarrow (\pi \underset{A_{11}}{\approx} \overset{\#}{\beta}) \Rightarrow \overset{\#}{\pi} \underset{A_{11}}{\approx} \overset{\#}{\beta}$. But $\pi^d \underset{A_{11}}{\approx} \overset{\#}{\pi}$ (since $d \notin L) \Rightarrow \pi^d \underset{A_{11}}{\approx} \beta \Rightarrow \pi^d \in A(\beta)$ & $\pi^d \notin \overset{\#}{\beta}$. Then the solution set of $x^d \in A(\beta)$ in A_{11} is $A(\overset{\#}{\beta})$.

Definition 1.36. Let $\beta \in [13]$ of S_{13}, where $\beta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13})$. We define classes $[13]^{\pm}$ of A_{13} by:

$A(\beta) = [13]^+ = \{\mu \in [13] \| \mu = t\beta t^{-1};$ *for some* $t \in A_{13}\}$ and

$A(\beta^{\#}) = [13]^- = \{\mu \in [13] \mid \mu = t\beta^{\#}t^{-1};$ for some $t \in A_{13}\}$.

where $\beta^{\#} = (a_1, a_3, a_5, a_7, a_9, a_{11}, a_{13}, a_2, a_4, a_6, a_8, a_{10}, a_{12})$.

Remark 1.37. (*i*) Let $\beta \in [13]$ of S_{13} where, $\beta_1 = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13})$

$\beta_1 = (a_1, a_3, a_5, a_7, a_9, a_{11}, a_{13}, a_2, a_4, a_6, a_8, a_{10}, a_{12})$
$\beta_2 = (a_1, a_4, a_7, a_{10}, a_{13}, a_3, a_6, a_9, a_{12}, a_2, a_5, a_8, a_{11})$
$\beta_3 = (a_1, a_5, a_9, a_{13}, a_4, a_8, a_{12}, a_3, a_7, a_{11}, a_2, a_6, a_{10})$
$\beta_4 = (a_1, a_6, a_{11}, a_3, a_8, a_{13}, a_5, a_{10}, a_2, a_7, a_{12}, a_4, a_9)$
$\beta_5 = (a_1, a_7, a_{13}, a_6, a_{12}, a_5, a_{11}, a_4, a_{10}, a_3, a_9, a_2, a_8)$

and d is a positive integer number we have:

(1)	$\beta^d = \beta \iff d \equiv 1 \pmod{13}$
(2)	$\beta^d = \beta_1 \iff d \equiv 2 \pmod{13}$
(3)	$\beta^d = \beta_2 \iff d \equiv 3 \pmod{13}$
(4)	$\beta^d = \beta_3 \iff d \equiv 4 \pmod{13}$
(5)	$\beta^d = \beta_4 \iff d \equiv 5 \pmod{13}$
(6)	$\beta^d = \beta_5 \iff d \equiv 6 \pmod{13}$
(7)	$\beta^d = \beta_5^{-1} \iff d \equiv 7 \pmod{13}$
(8)	$\beta^d = \beta_4^{-1} \iff d \equiv 8 \pmod{13}$
(9)	$\beta^d = \beta_3^{-1} \iff d \equiv 9 \pmod{13}$
(10)	$\beta^d = \beta_2^{-1} \iff d \equiv 10 \pmod{13}$
(11)	$\beta^d = \beta_1^{-1} \iff d \equiv 11 \pmod{13}$
(12)	$\beta^d = \beta^{-1} \iff d \equiv 12 \pmod{13}$

(*ii*) (1) $A(\beta) = A(\beta^{-1})$, $A(\beta_1) = A(\beta_1^{-1})$, $A(\beta_2) = A(\beta_2^{-1})$, $A(\beta_3) = A(\beta_3^{-1})$, $A(\beta_4) = A(\beta_4^{-1})$, and $A(\beta_5) = A(\beta_5^{-1})$ [Since for each $\lambda = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}) \in [13]^{\mp}$ in A_{13}, $\exists \mu = (b_1, b_{12})\ (b_2, b_{11})\ (b_3, b_{10})\ (b_4, b_9)\ (b_5, b_8)\ (b_6, b_7) \in A_9$ such that $\mu\lambda\mu^{-1} = \lambda^{-1}$].

(2) $A(\beta) = A(\beta_2)$ [Since $\exists t = (a_2, a_4, a_{10})\ (a_6, a_3, a_7)\ (a_{11}, a_5, a_{13})\ (a_8, a_9, a_{12}) \in A_{13}$ such that $t\beta t^{-1} = \beta_2$].

(3) $A(\beta) = A(\beta_3)$ [Since $\exists t = (a_6, a_8, a_3, a_9, a_7, a_{12})\ (a_{10}, a_{11}, a_2, a_5, a_4, a_{13}) \in A_{13}$ such that $t\beta t^{-1} = \beta_3$].

(4) $A(\beta_1) = A(\beta_4)$ [Since $\exists t = (a_7, a_3, a_6)\ (a_5, a_{11}, a_{13})\ (a_9, a_8, a_{12})\ (a_4, a_2, a_{10}) \in A_{13}$ such that $t\beta_1 t^{-1} = \beta_4$].

(5) $A(\beta_1) = A(\beta_5)$ [Since $\exists t = (a_2, a_4, a_{10})\ (a_6, a_3, a_7)\ (a_{11}, a_5, a_{13})\ (a_8, a_9, a_{12}) \in A_{13}$ such that $t\beta_1 t^{-1} = \beta_5$].

Theorem 1.38. *Let* $L = \{m \in N | m \equiv q (mod 13);$ *for some* $q = 1.3.4.9.10.12\}$. *If d is a positive integer number such that* $gcd(d, 13 = 1)$ *and* $\beta \in [13]$ *of* S_{13}, *then the solutions of* $x^d \in A(\beta)$ *in* A_{13} *are:*

(1) if $A(\beta)$ *if* $d \in L$.
(2) if $A(\overset{\#}{\beta})$ *if* $d \notin L$.

Proof. Since $\beta \in [13] \cap H \cap A_{13}$, $[13]$ splits into two classes $A(\beta)$ & $A(\overset{\#}{\beta})$ of A_{13} and since $gcd(d, 13) = 1$, d, does not divide 13. Then by Taban (2007, lemma 3.2) we have $[13] = A(\beta) \cup A(\overset{\#}{\beta})$ is a solution set of $x^d \in [13] = A(\beta) \cup A(\overset{\#}{\beta})$ in S_{13}. However, $A(\beta) \cap A(\overset{\#}{\beta}) = \phi$, so for each $\pi \in [13] \Rightarrow (\pi \in A(\beta)\ \&\ \pi \notin A(\overset{\#}{\beta}))$ or $(\pi \notin A(\beta)\ \&\ \pi \in A(\overset{\#}{\beta}))$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then we have $\pi \underset{A_{13}}{\approx} \beta$ (π conjugate to β in A_{13}). However, $\pi^d \underset{A_{13}}{\approx} \pi$ (since $d \in L) \Rightarrow \pi^d \underset{A_{13}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A(\overset{\#}{\beta})$. If $\pi \in A(\overset{\#}{\beta})$, we have $(\pi \underset{A_{13}}{\approx} \overset{\#}{\beta}$. But $\pi^d \underset{A_{13}}{\approx} \pi$ (since $(d \in L) \Rightarrow \pi^d \underset{A_{13}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A(\overset{\#}{\beta})\ \&\ \pi^d \notin A(\beta)$. Then the solution set of $x^d \in A(\beta)$ in A_{13} is $A(\beta)$.

(2) Assume $d \notin L$, if $\pi \in A(\beta)$, then we have $\pi \underset{A_{13}}{\approx} \beta \Rightarrow \overset{\#}{\pi} \underset{A_{13}}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_{13}}{\approx} \overset{\#}{\pi}$ (since $d \notin L) \Rightarrow \pi^d \underset{A_{13}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A(\overset{\#}{\beta}) \& \pi^d \notin A(\beta)$. If $\pi \in A(\overset{\#}{\beta})$, so $(\pi \underset{A_{13}}{\approx} \overset{\#}{\beta}) \Rightarrow \overset{\#}{\pi} \underset{A_{13}}{\approx} \beta$. But $\pi^d \underset{A_{13}}{\approx} \overset{\#}{\pi}$ (since $d \notin L) \Rightarrow \pi^d \underset{A_{13}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A(\overset{\#}{\beta})$. Then the solution set of $x^d \in A(\beta)$ in A_{13} is $A(\overset{\#}{\beta})$.

Lemma 1.39. *Let* $A(\beta)$ *be the conjugacy class of* β *in,* A_n, $n > 14$, *and* $\beta \in [k_1, k_2, \ldots, k_l] \cap H$, *where* $[k_1, k_2, \ldots, k_l] \in F_n$ *is a class of* S_n. *If p is a prime number such that* $gcd(p, k_i) = 1$, *for each i, then the solutions of* $x^p \in A(\beta)$ *are:*

(1) $[k_1, k_2, \ldots, k_l]^-$ *if* $\beta^p = (\beta^{-1}$ *or* $\gamma)$, *where* γ *is conjugate to* β^{-1}.
(2) $[k_1, k_2, \ldots, k_l]^+$ *if* $\beta^p = (\beta$ *or* $\gamma)$, *where* γ *is conjugate to* β.

Proof. $\beta \in A_n \cap H \cap [k_1, k_2, \ldots, k_l]$, $[k_1, k_2, \ldots, k_l]$, splits into two classes $[k_1, k_2, \ldots, k_l]^{\pm}$ of A_n, $[k_1, k_2, \ldots, k_l] \in F \Rightarrow A(\beta) = [k_1, k_2, \ldots, k_l]^+$, and $A(\beta^{-1}) = [k_1, k_2, \ldots, k_l]^{-1}$. So, since $gcd(p, k_i) = 1$ for each i. Then by Taban (2007, lemma 2.8) we have for each $\lambda \in [k_1, k_2, \ldots, k_l] \Rightarrow \lambda^p \in [k_1, k_2, \ldots, k_l]$.

(1) Assume $\beta^p = (\beta^{-1}$ or $\lambda = b\beta^{-1}b^{-1}$; for some $b \in A_n$), and let. $\lambda \in [k_1, k_2, \ldots, k_l]$. Then either $\lambda \in [k_1, k_2, \ldots, k_l]^+$ or $\lambda \in [k_1, k_2, \ldots, k_l]^-$.

If $\lambda \in [k_1, k_2, \ldots, k_l]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \left(\begin{array}{c} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{array}\right],$$

$\lambda^p \in [k_1, k_2, \ldots, k_l]^- \Rightarrow \lambda^p \notin [k_1, k_2, \ldots, k_l]^+ = A(\beta)$,

If $\lambda \in [k_1, k_2, \ldots, k_l]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{array}\right],$$

$\lambda^p \in [k_1, k_2, \ldots, k_l]^+ = A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[k_1, k_2, \ldots, k_l]^-$.

(2) Assume $\beta^p = (\beta$ or $\lambda = b\beta b^{-1}$ for some $b \in A_n$), and let $\lambda \in [k_1, k_2, \ldots, k_l]$. Then either $\lambda \in [k_1, k_2, \ldots, k_l]^+$ or $\lambda \in [k_1, k_2, \ldots, k_l]^-$.

If $\lambda \in [k_1, k_2, \ldots, k_l]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \left(\begin{array}{c} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{array}\right],$$

$\lambda^p \in [k_1, k_2, \ldots, k_l]^+ = A(\beta) \Rightarrow \lambda^p \notin [k_1, k_2, \ldots, k_l]^-$,

If $\lambda \in [k_1, k_2, \ldots, k_l]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \left(\begin{array}{c} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{array}\right],$$

$\lambda^p \in [k_1, k_2, \ldots, k_l]^- \Rightarrow \lambda^p \notin A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[k_1, k_2, \ldots, k_l]^+$.

Example 1.40. Let $\beta = \mu\lambda$, where $\mu = (1\ 2\ 3\ 4\ 5)$ and $\lambda = (6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16)$. Find the solutions of $x^{13} \in A(\beta)$ in A_{16}.

Solution:

since $n = 16 > 14$, $\beta \in [5,11] \cap H$, $[5,11] \in F_n$, $gcd(13,5) = 1$, $gcd(13,11) = 1$ and, $\exists \pi \in A_{16} \ni \pi\beta^{13}\pi^{-1} = \beta$ where $\beta^{13} = (1\ 4\ 2\ 5\ 3)\ (6\ 8\ 10\ 12\ 14\ 16\ 7\ 9\ 11\ 13\ 15)$ and $\pi = (3\ 5\ 4\ 2)\ (7\ 12\ 9\ 13\ 15\ 16\ 11\ 14\ 10\ 8)$. Then the solution set of $x^{13} \in A(\beta)$ in A_{16} is $[5,11]^+$.

Lemma 1.41. *Let $A(\beta)$ be the conjugacy class of β in A_n, $n > 14$, and, $\beta \in [k_1,k_2,\ldots,k_l] \cap H$ where $[k_1,k_2,\ldots,k_l] \in F_n$ is a class of. If p is a prime number such that $p | k_i$, for some i, then no solution of $x^p \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap H \cap [k_1,k_2,\ldots,k_l]$, $[k_1,k_2,\ldots,k_l]$, splits into two classes $[k_1,k_2,\ldots,k_l]^{\pm}$ of A_n and $[k_1,k_2,\ldots,k_l] \in F \Rightarrow A(\beta) = [k_1,k_2,\ldots,k_l]$. Also, since, then by Taban (2007, lemma 3.1) we have no solution for $x^p \in [k_i]$ in S_n. Then no solution for in S_n. So no solution for $x^p \in A(\beta)$ in A_n.

Lemma 1.42. *Let $A(\beta)$ be the conjugacy class of β in A_n, $n > 14$, and $\beta \in [k_1,k_2,\ldots,k_l] \cap H$, where $[k_1,k_2,\ldots,k_l] \in F_n$ is a class of S_n, p is a prime number, m is a positive integer. If for some $(1 \leqslant i \leqslant l)$ such that $p | k_i$, then no solution of $x^{p^m} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap H \cap [k_1,k_2,\ldots,k_l]$, $[k_1,k_2,\ldots,k_l]$ splits into two classes $[k_1,k_2,\ldots,k_l]^{\pm}$ of A_n and $[k_1,k_2,\ldots,k_l] \in F \Rightarrow A(\beta) = [k_1,k_2,\ldots,k_l]^+$. Also, since $p | k_i$, then by Taban (2007, lemma 3.12) we have no solution for $x^{p^m} \in [k_i]$ in S_n. Then no solution for $x^{p^m} \in [k_1,k_2,\ldots,k_l]$ in S_n, so no solution for $x^{p^m} \in A(\beta)$ in A_n.

The number of solution

If β is an even permutation and $\beta \in C^{\alpha}(\beta) \cap H$, where $C^{\alpha}(\beta)$ is a class of β in S_n, we have $C^{\alpha}(\beta)$ splits into two classes $C^{\alpha}(\beta)^{\pm}$ of equal order $\Rightarrow A(\beta) = C^{\alpha}(\beta)^+$ or $C^{\alpha}(\beta)^-$, where $A(\beta)$ is a class of β in A_n. If $C^{\alpha}(\beta)^+$ or $C^{\alpha}(\beta)^-$ is a solution in A_n of any class equation in A_n, then the number of solutions is the number of all the elements that belong to the class $C^{\alpha}(\beta)^+$ or $C^{\alpha}(\beta)^-$. However, $| C^{\alpha}(\beta)^+ | = | C^{\alpha}(\beta)^- | = \frac{|C^{\alpha}(\beta)|}{2}$. So the number of the solutions for the class equation $x^d = \beta$ in A_n is only $\frac{n!}{2z_{\alpha}}$.

Example 1.43. Find the solutions of $x^p \in A(234)$ in A_4 and the number of the solutions

(i) if $p = 13$.
(ii) if $p = 17$.

Solution:

$n = 4 \Rightarrow \beta = (2\quad 3\quad 4) \in [1,3]$. However, $[1,3] \subset H \Rightarrow \beta \in [1,3] \cap H$. Now we show that:

(i) If $p = 13$, then we have $gcd(13,3) = 1$, $gcd(13,1) = 1$, and $(2\ 3\ 4)^{13} = (2\ 3\ 4)$. Then by (1.12) the solution of $x^{13} \in A(2\ 3\ 4)$ in A_4 is $[1,3]^+$ and the number of the solution is $\frac{|[1,3]|}{2} = \frac{4!}{2\times 3} = 4$ permutations, where $[1,3]^- = \{(1\ 3\ 2), (2\ 3\ 4), (1\ 4\ 3), (1\ 2\ 4)\}$.

(ii) If $p = 17$, then we have $gcd(17,3) = 1$, $gcd(17,1) = 1$, and $(2\ 3\ 4)^{17} = (4\ 3\ 2) = (2\ 3\ 4)^{-1}$. Then by (1.12) the solution of $x^{17} \in\ \in A(2\ 3\ 4)$ in A_4 is $[1,3]^-$ and the number of the solution is $\frac{|[1,3]|}{2} = \frac{4!}{2\times 3} = 4$ permutations, where $[1,3]^- = \{(1\ 2\ 3), (2\ 4\ 3), (1\ 3\ 4), (1\ 4\ 2)\}$.

Example 1.44. Find the solutions of $x^{15} \in A(\beta)$ in A_7 and the number of the solutions where $\beta = (2\ 4\ 1\ 3\ 5\ 7\ 6)$.

Solution:

$n = 7 \Rightarrow \beta = (2\quad 4\quad 1\quad 3\quad 5\quad 7\quad 6) \in [7]$. However, $[7] \subset H \Rightarrow \beta \in [7] \cap H$. Assume $p = 3$, and $q = 5$, we have $gcd(3,7) = 1$ $gcd(5,7) = 1$, and $\beta^{15} = \beta$. Then by (1.15) the solution set of $x^{15} \in A(\beta)$ in A_7 is $[7]^+ = A(\beta)$ and the number of the solutions is $\frac{7!}{2\times 7} = 360$ permutations.

Example 1.45. Find the solutions of $x^{14} \in A((4\ 1\ 3)\ (2\ 6\ 7\ 5\ 8))$ in A_8 and the number of the solutions.

Solution:

$n = 8 \Rightarrow \beta = (4\ 1\ 3)(2\ 6\ 7\ 5\ 8) \in [3,5]$. However, $[3,5] \subset H \Rightarrow \beta \in [3,5] \cap H$. Let $p = 14$, then we have gcd(11,3) = 1, gcd(11,5) = 1, and $\beta^{14} = (1 4 3)(8 5 7 6 2) = \beta^{-1}$. Then by (1.12) the solution of $x^{14} \in A((4\ 1\ 3)(2\ 6\ 7\ 5\ 8))$ in A_8 is $[3,5]^-$ and the number of the solution is $\frac{8!}{2\times 3\times 5\times} = 1344$ permutations.

2. Conclusions

By the Cayley's theorem: Every finite group G is isomorphic to a subgroup of the symmetric group S_n, for some $n \geqslant 1$. Then we can discuss these propositions. Let $x^d = g$ be class equation in finite group G and assume that $f{:}G \cong A_n$, for some $n \notin \theta$ and $f(g) \in H \cap C^{\alpha}$. The first question we are concerned with is: what is the possible value of d provided that there is no solution for $x^d = g$ in G? The second question we are concerned with is: what is the possible value of d provided that there is a solution for $x^d = g$ in G? and then we can find the solution and the number of the solution for $x^d = g$ in G by using Cayley's theorem and our theorems in this paper. In another direction, let G be a finite group, and $\pi_i(G) = \{g \in G | \, i$ the least positive integer number satisfying $g^i = 1\}$. If $| \pi_i(G) | = k_i$, then we write $\pi_i(G) = \{g_{i1}, g_{i2}, \ldots, g_{ik_i}\}$, and $\prod = \{\pi_i(G)\}_{i \geqslant 1}$. For each $g \in G$ and $g_{ij} \in \pi_i(G)$ we have $(gg_{ij}g^{-1})^i = 1$. By the Cayley's theorem we can suppose that (f: $G \cong S_n$) or (f: $G \cong A_n$). Also the questions can be summarized as follows:

(1) Is $\prod = \{\pi_i(G)\}_{i \geqslant 1}$ collection set of conjugacy classes of G?
(2) Is there some $i \geqslant 1$, such that $f^{-1}(C^{\alpha}) = \pi_i(G)$, for each $C^{\alpha}$of A_n, where (f: $G \cong A_n$)?
(3) Is there some $i \geqslant 1$, such that $f^{-1}(C^{\alpha}) = \pi_i(G)$, for each $C^{\alpha}$of S_n, where (f : $G \cong S_n$)?
(4) If ($G \cong S_n$) and $p(n)$ is the number of partitions of n, is $| \prod | = p(n)$?
(5) If ($G \cong A_n$) and A_n has m ambivalent conjugacy classes. It is true that is also necessarily G has m ambivalent conjugacy classes?

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