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تحليلات التغاير الكهرو-المرن-اللزج لمسألة إتصال في وجود احتكاك

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الملخص:

تم في هذا البحث دراسة نموذج رياضي لوصف اتصال إحتكاكي شبه سكوني بين جسم كهرواجهادي وأساس قابل لتغير الشكل. لقد تم استخدام قانون البناء الكهربائي المرن–اللزج الغير خطي لنمذجة المواد ذات الخواص الكهرواجهاديه. إن الإتصال تم وصفه بواسطة حالة الإمتثال العادي وصيغة لقانون كولوم للاحتكاك. كما تم إشتقاق صياغة تغايرية للنموذج على شكل نظام متوافق الإزاحة والجهد الكهربائي. كما تم تأسيس وجود حل متفرد ضعيف للنموذج بناءً على فرضية صغر معامل الإحتكاك. لقد اعتمد البرهان على حجج تباينات التغاير التطورية ومشغلات النقاط الثابتة.

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REVIEW ARTICLE

Variational analysis of an electro viscoelastic contact problem with friction

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Abstract A mathematical model which describes the quasistatic frictional contact between a piezoelectric body and a deformable foundation is studied in this paper. A nonlinear electro-viscoelastic constitutive law is used to model the piezoelectric material. The contact is described with the normal compliance condition and a version of Coulomb's law of friction. A variational formulation of the model, in the form of a coupled system for the displacements and the electric potential, is derived. The existence of a unique weak solution of the model is established under a smallness assumption of the friction coefficient. The proof is based on arguments of evolutionary variational inequalities and fixed points of operators.

1. Introduction

In this work, we continue the research of Lerguet et al. (2007) with a perfect insulator foundation and an other version of Coulomb's low friction. We formulate and analyze the variational formulation of the electro-viscoelastic problem. Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or complex metal forming processes are just a few examples. Because of the importance of contact processes in structural and mechanical systems, a considerable effort has been made in its modeling and numerical simulations and so, the engineering literature concerning this topic is rather extensive (Berdichevsky, 2009; Muradova and Stavroulakis, 2007; Fisher-Cripps, 2000).

There is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials (Bisenga et al., 2002; Sofonea and Essoufifi, 2004; Drabla and Zellagui, 2009, 2011). Indeed, many actuators and sensors in engineering controls are made of piezoelectric ceramics. However, there exists virtually no mathematical results about contact problems for such materials and there is a need to expand the MTCM (Mathematical Theory of Contact Mechanics) to include the coupling between the mechanical and electrical material properties.

The piezoelectric effect is characterized by such a coupling between the mechanical and electrical properties of the materials. This coupling, leads to the appearance of electric field in the presence of a mechanical stress, and conversely, mechanical stress is generated when electric potential is applied. The first effect is used in sensors, and the reverse effect is used in actuators.

On a nano-scale, the piezoelectric phenomenon arises from a nonuniform charge distribution within a crystal's unit cell.

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When such a crystal is deformed mechanically, the positive and negative charges are displaced by a different amount causing the appearance of electric polarization. So, while the overall crystal remains electrically neutral, an electric polarization is formed within the crystal. This electric polarization due to mechanical stress is called *piezoelectricity*. A deformable material which exhibits such a behavior is called a *piezoelectric material*. Piezoelectric materials for which the mechanical properties are elastic are also called *electro-elastic materials* and piezoelectric materials for which the mechanical properties are viscoelastic are also called *electro-viscoelastic materials*.

Only some materials exhibit sufficient piezoelectricity to be useful in applications. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate, and polyvinylidene fluoride (a polymer film), and are used extensively as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and in measuring equipment. General models for electro-elastic materials can be found in Mindlin (1968), Mindlin (1972) and, more recently, in Ikeda (1990). A static and a slip-dependent frictional contact problems for electro-elastic materials were studied in Bisenga et al. (2002). A contact problem with normal compliance for electro-viscoelastic materials was investigated in Sofonea et al. (2004). In the last two references the foundation was assumed to be insulated. The variational formulations of the corresponding problems were derived and existence and uniqueness of weak solutions were obtained.

We present in this work two logically connected aspects of the theory of *electro-viscoelastic materials*: the constitutive theory and the variational formulation of the related initial boundary value problem.

The paper is structured as follows. In Section 2 we describe the model of the frictional contact process between an electro-viscoelastic body and a deformable foundation. In Section 3 we introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. It consists of a variational inequality for the displacement field coupled with a nonlinear time-dependent variational equation for the electric potential. We state our main result, the existence of a unique weak solution to the model in Theorem 3.1. The proof of the theorem is provided in Section 4, where it is carried out in several steps and is based on arguments of evolutionary inequalities with monotone operators, and a fixed point theorem.

2. The model

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal $\boldsymbol{\nu}$. The body is acted upon by body forces of density $\boldsymbol{f}_0$ and has volume free electric charges of density $\boldsymbol{q}_0$. It is also constrained mechanically and electrically on the boundary. To describe these conditions, we assume a partition of Γinto three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\,\Gamma_1 > 0$ and $meas\,\Gamma_a > 0$; these conditions allow the use of coercivity arguments which guarantee the uniqueness of the solution for the model. The body is clamped on Γ_1 and, therefore, the displacement field $u = (u_1, \ldots, u_d)$ vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface free electrical charge of density $\boldsymbol{q}_2$ is prescribed on Γ_b. In the reference configuration the body may come in contact over Γ_3 with an insulator obstacle, which is also called the foundation. The contact is frictional and is modeled with the normal compliance condition and a version of Coulomb's law of friction. Also, there may be electrical charges on the part of the body which is in contact with the foundation and which vanish when contact is lost. We are interested in the evolution of the deformation of the body and of the electric potential on the time interval $[0, T]$. The process is assumed to be isothermal, electrically static, i.e., all radiation effects are neglected, and mechanically quasistatic; i.e., the inertial terms in the momentum balance equations are neglected. We denote by $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variables, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on $\boldsymbol{x}$ and t. In this paper $i, j, k, l = 1, \ldots, d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of $\boldsymbol{x}$. A dot over a variable represents the time derivative. We use the notation $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ and "$\cdot$" and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i$, $\|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$, and $\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $\|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$ for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, by $\boldsymbol{u}_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, $\sigma_\nu = \sigma_{ij}\ \nu_i \nu_j$, and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. The classical model for the process is as follows.

Problem $\mathcal{P}$. *Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$ and an electric displacement field $\mathbf{D} : \Omega \times [0, T] \to \mathbb{R}^d$ such that*

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{G}\varepsilon(\boldsymbol{u}) - \mathcal{E}^*\mathbf{E}(\varphi) \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \mathcal{B}\mathbf{E}(\varphi) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$\mathrm{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \mathbf{0} \quad \text{in } \Omega \times (0, T), \tag{2.3}$$

$$\mathrm{div}\, \mathbf{D} - \boldsymbol{q}_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.4}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0, T), \tag{2.5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.6}$$

$$-\boldsymbol{\sigma}_\nu = p_\nu(\boldsymbol{u}_\nu - \boldsymbol{g}) \quad \text{on } \Gamma_3, \times (0, T), \tag{2.7}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \leqslant \mu p_\nu(\boldsymbol{u}_\nu - \boldsymbol{g}); \quad \text{on } \Gamma_3 \times (0, T), \\ \|\boldsymbol{\sigma}_\tau\| < \mu p_\nu(\boldsymbol{u}_\nu - \boldsymbol{g}) \Rightarrow \ \dot{\boldsymbol{u}}_\tau = 0; \\ \|\boldsymbol{\sigma}_\tau\| = \mu p_\nu(\boldsymbol{u}_\nu - \boldsymbol{g}) \Rightarrow \quad \text{there exists } \lambda \geqslant 0 \\ \qquad\qquad\qquad \text{such that } \boldsymbol{\sigma}_\tau = -\lambda \dot{\boldsymbol{u}}_\tau, \end{cases} \tag{2.8}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0, T), \tag{2.9}$$

$$\mathbf{D} \cdot \boldsymbol{\nu} = \boldsymbol{q}_2 \quad \text{on } \Gamma_b \times (0, T), \tag{2.10}$$

$$\mathbf{D} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_3 \times (0, T), \tag{2.11}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{2.12}$$

We now describe problem (2.1)–(2.12) and provide explanation of the equations and the boundary conditions.

First, Eqs. (2.1) and (2.2) represent the nonlinear electro viscoelastic constitutive law in which $\boldsymbol{\sigma} = (\sigma_{ij})$ is the stress tensor, ε(u) denotes the linearized strain tensor, $\mathcal{A}$ and $\mathcal{G}$ are the viscosity and elasticity operators, respectively, $\mathcal{E} = (e_{ijk})$

represents the third-order piezoelectric tensor, $\mathcal{E}^*$ is its transpose, $\mathcal{B} = (b_{ij})$ denotes the electric permittivity tensor, and $\mathbf{D} = (D_1, \ldots, D_d)$ is the electric displacement vector. Since we use the electrostatic approximation, the electric field satisfies $\mathbf{E}(\varphi) = -\$\ \varphi$, where φ is the electric potential.

We recall that $\boldsymbol{\varepsilon}(\mathrm{u}) = (\boldsymbol{\varepsilon}_{ij}(\mathrm{u}))$ and $\boldsymbol{\varepsilon}_{ij}(\mathrm{u}) = (u_{i,j} + u_{j,i})/2$. The tensors $\mathcal{E}$ and $\mathcal{E}^*$ satisfy the equality

$$\mathcal{E}\boldsymbol{\sigma}\cdot\boldsymbol{v} = \boldsymbol{\sigma}\cdot\mathcal{E}^*\boldsymbol{v} \quad \forall\boldsymbol{\sigma} = (\sigma_{ij}) \in \mathbb{S}^d,\ \boldsymbol{v} \in \mathbb{R}^d$$

and the components of the tensor $\mathcal{E}^*$ are given by $e^*_{ijk} = e_{kij}$.

A viscoelastic Kelvin–Voigt constitutive relation (Han and Sofonea, 2002) is given in (2.1), in which the dependence of the stress on the electric field is taken into account. Relation (2.2) describes a linear dependence of the electric displacement field $\mathbf{D}$ on the strain and electric fields; such a relation has been frequently employed in the literature (Bisenga et al., 2002).

Next, Eqs. (2.3) and (2.4) are the steady equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, i.e.,

$$\mathrm{Div}\ \boldsymbol{\sigma} = (\sigma_{ij,j}), \quad \mathrm{div}\ \mathbf{D} = (D_{i,i}).$$

We use these equations since the process is assumed to be mechanically quasistatic and electrically static.

Conditions (2.5) and (2.6) are the displacement and traction boundary conditions, whereas (2.9) and (2.10) represent the electric boundary conditions; the displacement field and the electrical potential vanish on Γ_1 and Γ_a, respectively, while the forces and free electric charges are prescribed on Γ_2 and Γ_b, respectively. Finally, the initial displacement u_0 in (2.12) is given.

We turn to the boundary conditions (2.7), (2.8), and (2.11) which describe the contact on the surface Γ_3 and in which our main interest is. First, the normal compliance function p_ν, in (2.7), is described below, and $\boldsymbol{g}$ represents the gap in the reference configuration between Γ_3 and the foundation, measured along the direction of $\boldsymbol{\nu}$. When positive, $u_\nu - g$ represents the interpenetration of the surface asperities into those of the foundation. This condition was first used in a large number of papers (Kikuchi and Oden, 1988; Han and Sofonea, 2002; Klarbring et al., 1988). Condition (2.8) is the associated friction law where μp_ν is a given function. According to (2.8) the tangential shear cannot exceed the maximum frictional resistance $\mu p_\nu(\mathrm{u}_\nu - \boldsymbol{g})$, the so-called friction bound. Moreover, when sliding commences, the tangential shear reaches the friction bound and opposes the motion. Frictional contact conditions of the form (2.7) and (2.8) have been used in various papers (Rochdi et al., 1998; Han and Sofonea, 2002).

Next, (2.11) is the electrical contact condition on Γ_3 which decouples the electrical and mechanical problems on the contact surface. Condition (2.11) models the case when the obstacle is a perfect insulator and was used in Bisenga et al., 2002 and Sofonea et al., (2004). Now, we derive in the next section a variational formulation of the problem and investigate its solvability. Moreover, variational formulations are also starting points for the construction of finite element algorithms for this type of problems.

3. Variational formulation and the main result

We use the standard notation for the L^p and the Sobolev spaces associated with $\Omega$and Γ and, for a function $\psi \in H^1(\Omega)$ we still write ψ to denote its trace on Γ. We recall that the summation convention applies to a repeated index.

For the electric displacement field we use two Hilbert spaces

$$\mathcal{W} = L^2(\Omega)^d, \quad \mathcal{W}_1 = \{\mathbf{D} \in \mathcal{W} : \mathrm{div}\ \mathbf{D} \in L^2(\Omega)\}$$

endowed with the inner products

$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_\Omega D_i E_i dx, \quad (\mathbf{D}, \mathbf{E})_{\mathcal{W}_1} = (\mathbf{D}, \mathbf{E})_{\mathcal{W}} + (\mathrm{div}\mathbf{D}, \mathrm{div}\mathbf{E})_{L^2(\Omega)}$$

and the associated norms $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_{\mathcal{W}_1}$, respectively. The electric potential field is to be found in

$$W = \{\psi \in H^1(\Omega) : \psi = 0 \quad \text{on } \Gamma_a\}.$$

Since $meas\,\Gamma_a > 0$, the Friedrichs–Poincaré inequality holds, thus,

$$\|\nabla\psi\|_{\mathcal{W}} \geqslant c_F\|\psi\|_{H^1(\Omega)} \quad \forall\psi \in W, \tag{3.1}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product

$$(\varphi, \psi)_W = (\nabla\varphi, \nabla\psi)_{\mathcal{W}}$$

and let $\|\cdot\|_W$ be the associated norm. It follows from (3.1) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and therefore $(W,\|\cdot\|_W)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant c_0, depending only on Ω, Γ_a and Γ_3, such that

$$\|\psi\|_{L^2(\Gamma_3)} \leqslant c_0\|\psi\|_W \quad \forall\psi \in W. \tag{3.2}$$

We recall that when $\mathbf{D} \in \mathcal{W}_1$ is a sufficiently regular function, the Green type formula holds:

$$(\mathbf{D}, \nabla\psi)_{L^2(\Omega)^d} + (\mathrm{div}\ \mathbf{D}, \psi)_{\mathcal{W}} = \int_\Gamma \mathbf{D}\cdot\boldsymbol{\nu}\psi da \quad \forall\psi \in H^1(\Omega). \tag{3.3}$$

For the stress and strain variables, we use the real Hilbert spaces

$$Q = \{\boldsymbol{\tau} = (\tau_{ij}) : \ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega)^{d\times d}_{sym}, \quad Q_1 = \{\boldsymbol{\sigma} = (\sigma_{ij}) \in Q : \mathrm{div}\ \boldsymbol{\sigma} = (\sigma_{ij,j}) \in \mathcal{W}\}$$

endowed with the respective inner products

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}\tau_{ij}dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\mathrm{div}\ \boldsymbol{\sigma}, \mathrm{div}\ \boldsymbol{\tau})_{\mathcal{W}}$$

and the associated norms $\|\cdot\|_Q$ and $\|\cdot\|_{Q_1}$. For the displacement variable we use the real Hilbert space

$$H_1 = \{u = (u_i) \in \mathcal{W} : \boldsymbol{\varepsilon}(\boldsymbol{u}) \in Q\}$$

endowed with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_{\mathcal{W}} + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$$

and the norm $\|\cdot\|_{H_1}$.

When $\boldsymbol{\sigma}$ is a regular function, the following Green's type formula holds,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathrm{Div}\ \boldsymbol{\sigma}, \boldsymbol{v})_{L^2(\Omega)^d} = \int_\Gamma \boldsymbol{\sigma}v\cdot\boldsymbol{\nu}da \quad \forall\ \boldsymbol{v} \in H_1. \tag{3.4}$$

Next, we define the space

$$V = \{\boldsymbol{v} \in H_1 : \boldsymbol{v} = \mathbf{0} \quad \text{on } \Gamma_1\}.$$

Since $meas\,\Gamma_1 > 0$, Korn's inequality holds and

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \geqslant c_K\|\boldsymbol{v}\|_{H_1} \quad \forall \boldsymbol{v} \in V, \tag{3.5}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V we use the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$$

and let $\|\cdot\|_V$ be the associated norm. It follows from (3.5) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V and, therefore, the space $(V,(\cdot,\cdot)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\tilde{c}_0$, depending only on Ω, Γ_1 and Γ_3, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leqslant \tilde{c}_0\|\boldsymbol{v}\|_V \quad \forall \boldsymbol{v} \in V. \tag{3.6}$$

Finally, for a real Banach space $(X,\|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \leqslant p \leqslant \infty$, $k = 1, 2, \ldots$; we also denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the respective norms

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]}\|x(t)\|_X, \quad \|x\|_{C^1([0,T];X)} = \max_{t\in[0,T]}\|x(t)\|_X + \max_{t\in[0,T]}\|\dot{x}(t)\|_X.$$

Recall that the dot represents the time derivative.

We now list the assumptions on the problem's data. The *viscosity operator* $\mathcal{A}$ and the *elasticity operator* $\mathcal{G}$ are assumed to satisfy the conditions:

$$\begin{cases} \text{(a)} & \mathcal{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d, \\ \text{(b)} & \text{There exists } L_{\mathcal{A}}>0 \text{ such that} \\ & \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_2)\| \leqslant L_{\mathcal{A}}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\| \\ & \forall\, \boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{S}^d, \text{a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c)} & \text{There exists } m_{\mathcal{A}}>0 \text{ such that} \\ & (\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_2))\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2) \geqslant \\ & m_{\mathcal{A}}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|^2\ \forall\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{S}^d, \text{a.e. } \boldsymbol{x}\in\Omega. \\ \text{(d)} & \text{The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue} \\ & \text{measurable on } \Omega, \text{ for any } \boldsymbol{\xi}\in\mathbb{S}^d. \\ \text{(e)} & \text{The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } Q. \end{cases} \tag{3.7}$$

$$\begin{cases} \text{(a)} & \mathcal{G}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d. \\ \text{(b)} & \text{There exists } L_{\mathcal{G}}>0 \text{ such that} \\ & \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi}_1)-\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi}_2)\| \leqslant L_{\mathcal{G}}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\| \\ & \forall\, \boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{S}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c)} & \text{The mapping } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{\xi}) \text{ is measurable on } \Omega, \\ & \text{for any } \boldsymbol{\xi}\in\mathbb{S}^d. \\ \text{(d)} & \text{The mapping } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } Q. \end{cases} \tag{3.8}$$

The *piezoelectric tensor* $\mathcal{E}$ and the *electric permittivity tensor* $\mathcal{B}$ satisfy

$$\begin{cases} \text{(a)} & \mathcal{E}:\Omega\times\mathbb{S}^d\to\mathbb{R}^d. \\ \text{(b)} & \mathcal{E}(\boldsymbol{x},\boldsymbol{\tau}) = (e_{ijk}(\boldsymbol{x})\tau_{jk}) \quad \forall\boldsymbol{\tau}=(\tau_{ij})\in\mathbb{S}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c)} & e_{ijk}=e_{ikj}\in L^\infty(\Omega). \end{cases} \tag{3.9}$$

$$\begin{cases} \text{(a)} & \mathcal{B}:\Omega\times\mathbb{R}^d\to\mathbb{R}^d. \\ \text{(b)} & \mathcal{B}(\boldsymbol{x},\mathbf{E}) = (b_{ij}(\boldsymbol{x})E_j) \quad \forall\mathbf{E}=(E_i)\in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c)} & b_{ij}=b_{ji}\in L^\infty(\Omega). \\ \text{(d)} & \text{There exists } m_{\mathcal{B}}>0 \text{ such that } b_{ij}(\mathbf{x})E_iE_j \geqslant m_{\mathcal{B}}\|\mathbf{E}\|^2 \\ & \forall\mathbf{E}=(E_i)\ \in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \end{cases} \tag{3.10}$$

In linearized electro viscoelasticity, the constitutive laws (2.1) and (2.2) read

$$\sigma_{ij} = a_{ijkl}\varepsilon_{k,l}(\dot{\mathbf{u}}) + g_{ijkl}\varepsilon_{kl}(\mathbf{u}) - e_{kij}\varphi_{,k}, \quad D_i = e_{ijk}\varepsilon_{jk}(\mathbf{u}) + \beta_{ij}\varphi_{,j},$$

where a_{ijkl}, g_{ijkl}, β_{ij} are the components of the tensors $\mathcal{A}$, $\mathcal{G}$ and $\boldsymbol{\beta}$, respectively, and $\varphi_{,j} = \partial\varphi/\partial x_j$. Clearly, assumption (3.7) is satisfied if all the components a_{ijkl} belong to $L^\infty(\Omega)$ and satisfy the usual properties of symmetry and ellipticity:

$$a_{ijkl} = a_{jikl} = a_{klij}$$

and

$$a_{ijkl}\zeta_{ij}\zeta_{kl} \geqslant m_0\|\zeta\|^2$$

for $m_0 > 0$ and all symmetric tensors ζ. Assumption (3.8) is satisfied if g_{ijkl} belong to $L^\infty(\Omega)$ and satisfies the same symmetry properties.

A second example is provided by the nonlinear electro viscoelastic constitutive law,

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \alpha(\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{P}_K(\boldsymbol{\varepsilon}(\mathbf{u})) - \mathcal{E}^*\mathbf{E}(\varphi),$$

$$D_i = e_{ijk}\varepsilon_{jk}(\mathbf{u}) + \beta_{ij}\varphi_{,j},$$

Here $\mathcal{A}$ is a nonlinear fourth-order viscosity tensor that satisfies (3.7), α is a positive coefficient, K is a closed convex subset of $\mathbb{S}^d$ such that $0 \in K$ and $\mathcal{P}_K : \mathbb{S}^d \to K$ denotes the projection operator. Since the projection operator is nonexpansive, the elasticity operator $\mathcal{G}(x,\boldsymbol{\varepsilon}) = \alpha(\boldsymbol{\varepsilon} - \mathcal{P}_K\boldsymbol{\varepsilon})$ satisfies the condition (3.8).

The *normal compliance function* p_ν satisfies

$$\begin{cases} \text{(a)} & p_\nu:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+ \\ \text{(b)} & \exists L_\nu>0 \text{ such that } |p_\nu(\boldsymbol{x},\boldsymbol{u}_1)-p_\nu(\boldsymbol{x},u_2)| \leqslant L_\nu|u_1-u_2| \\ & \forall\, u_1,u_2\ \in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \\ \text{(c)} & \boldsymbol{x}\mapsto p_\nu(\boldsymbol{x},u) \text{ is measurable on } \Gamma_3, \text{ for all } u\in\mathbb{R}. \\ \text{(d)} & \boldsymbol{x}\mapsto p_\nu(\boldsymbol{x},u)=0, \text{ for all } u\leqslant 0. \end{cases} \tag{3.11}$$

An example of a normal compliance function p_ν which satisfies conditions (3.11) is $p_\nu(\boldsymbol{u}) = c_\nu\boldsymbol{u}_+$ where $c_\nu \in L^\infty(\Gamma_3)$ is a positive surface stiffness coefficient, and $\boldsymbol{u}_+ = \max\{0,\boldsymbol{u}\}$.

The forces, tractions, volume and surface free charge densities satisfy

$$\boldsymbol{f}_0 \in W^{1,p}(0,T;L^2(\Omega)^d), \tag{3.12}$$

$$\boldsymbol{f}_2 \in W^{1,p}(0,T;L^2(\Gamma_2)^d), \tag{3.13}$$

$$\boldsymbol{q}_0 \in W^{1,p}(0,T;L^2(\Omega)), \tag{3.14}$$

$$\boldsymbol{q}_2 \in W^{1,p}(0,T;L^2(\Gamma_b)). \tag{3.15}$$

Here, $1 \leqslant p \leqslant \infty$. Finally, we assume that the gap function, the friction coefficient and the initial displacement satisfy

$$\boldsymbol{g} \in L^2(\Gamma_3), \quad \boldsymbol{g} \geqslant 0 \quad \text{a.e. on } \Gamma_3, \tag{3.16}$$

$$\mu \in L^\infty(\Gamma_3), \quad \mu(\boldsymbol{x}) \geqslant 0 \quad \text{a.e. on } \Gamma_3 \tag{3.17}$$

$$\boldsymbol{u}_0 \in V. \tag{3.18}$$

Next, we define the three mappings $j_{fr} : V \times V \to \mathbb{R}$, $\boldsymbol{f} : [0,T] \to V$ and $\boldsymbol{q} : [0,T] \to W$, respectively, by

$$j_{fr}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} p_\nu(\boldsymbol{u}_\nu - \boldsymbol{g})\boldsymbol{v}_\nu \, da + \int_{\Gamma_3} \mu p_\nu(\boldsymbol{u}_\nu - \boldsymbol{g})\|\boldsymbol{v}_\tau\| da, \tag{3.19}$$

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot \boldsymbol{v}dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot \boldsymbol{v}da, \tag{3.20}$$

$$(\boldsymbol{q}(t),\psi)_W = \int_\Omega \boldsymbol{q}_0(t)\psi dx - \int_{\Gamma_b} \boldsymbol{q}_2(t)\psi da \tag{3.21}$$

for all $\boldsymbol{u},\boldsymbol{v} \in V$, φ, $\psi \in W$ and $t \in [0,T]$. We note that the definitions of, $\boldsymbol{f}$ and $\boldsymbol{q}$ are based on the Riesz representation theorem, moreover, it follows from assumptions (3.11)–(3.15) that the integrals in (3.19)–(3.21) are well-defined.

Using Green's formulas (3.3) and (3.4), it is easy to see that if $(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\varphi},\mathbf{D})$ are sufficiently regular functions which satisfy (2.3)–(2.10) then

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j_{fr}(\boldsymbol{u}(t),\boldsymbol{v}) - j_{fr}(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \geqslant (\boldsymbol{f}(t),\boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V, \tag{3.22}$$

$$(\mathbf{D}(t),\nabla\psi)_{\mathcal{W}} + (\boldsymbol{q}(t),\psi)_W = 0 \tag{3.23}$$

for all $\boldsymbol{v} \in V$, $\psi \in W$ and $t \in [0,T]$. We substitute (2.1) in (3.22), (2.2) in (3.23), note that $\mathbf{E}(\varphi) = -\$\varphi$, use the initial condition (2.12) and derive a variational formulation of Problem $\mathcal{P}$. It is in the terms of displacement and electric potential fields.

Problem $\mathcal{P}_V$. Find a displacement field $u : [0,T] \to V$ and an electric potential $\varphi : [0,T] \to W$ such that

$$\begin{cases} (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \\ +(\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j_{fr}(\boldsymbol{u}(t),\boldsymbol{v}) - j_{fr}(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \\ \geqslant (\boldsymbol{f}(t),\boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \end{cases} \tag{3.24}$$

for all $v \in V$ and $t \in [0,T]$,

$$(\mathcal{B}\nabla\varphi(t),\nabla\psi)_{\mathcal{W}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\nabla\psi)_{\mathcal{W}} = (\boldsymbol{q}(t),\psi)_W \tag{3.25}$$

for all $\psi \in W$ and $t \in [0,T]$, and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{3.26}$$

Next, we use (3.19) and (3.11)(b), keeping in mind (3.6), we obtain

$$j_{fr}(\boldsymbol{u}_1,\boldsymbol{v}_1) - j_{fr}(\boldsymbol{u}_1,\boldsymbol{v}_2) + j_{fr}(\boldsymbol{u}_2,\boldsymbol{v}_1) - j_{fr}(\boldsymbol{u}_2,\boldsymbol{v}_2) \leqslant \tilde{c}_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V. \tag{3.27}$$

now, by using (3.11)(b) and (3.17), it follows that the integral in (3.19) is well defined. Moreover, we have

$$j_{fr}(\boldsymbol{u},\boldsymbol{v}) \leqslant \tilde{c}_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)}\|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_V. \tag{3.28}$$

The inequalities (3.27) and (3.28) will be used in various places in the rest of the paper.

Our main existence and uniqueness result that we state now and prove in the next section is the following.

Theorem 3.1. *Assume that 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, hold. Then, there exists $\mu_0 > 0$ depending only on $\Omega,\Gamma_1,\Gamma_3,\mathcal{A}$, and p_ν such that, if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then Problem $\mathcal{P}_V$ has a unique solution (u,φ). Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{2,p}(0,T;V), \tag{3.29}$$

$$\varphi \in W^{1,p}(0,T;V). \tag{3.30}$$

It is easy to check that $\boldsymbol{\sigma}$ and $\mathbf{D}$ the function given by (2.1) and (2.2) respectively, satisfy:

$$\boldsymbol{\sigma} \in W^{1,p}(0,T;Q_1), \tag{3.31}$$

$$\mathbf{D} \in W^{1,p}(0,T;\mathcal{W}_1). \tag{3.32}$$

We conclude that the weak solution $(u,\sigma,\varphi,\mathbf{D})$ of the piezoelectric contact Problem $\mathcal{P}$ has the regularity implied in (3.29)–(3.31) and (3.32).

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, and consider the problem of finding $u : [0,T] \to X$ such that

$$\begin{cases} (A\dot{u}(t), v - \dot{u}(t))_X + (Bu(t), v - \dot{u}(t))_X + j(u(t),v) \\ -j(u(t),\dot{u}(t)) \geqslant (f(t), v - \dot{u}(t))_X \quad \forall v \in X, \ t \in [0,T], \end{cases} \tag{4.1}$$

$$u(0) = u_0. \tag{4.2}$$

To study problem (4.1) and (4.2) we need the following assumptions: The operator $A : X \to X$ is strongly monotone and Lipschitz continuous, i.e.,

$$\begin{cases} \text{(a)} & \text{There exists } m_A > 0 \text{ such that} \\ & (Au_1 - Au_2, u_1 - u_2)_X \geqslant m_A\|u_1 - u_2\|_X^2 \\ & \forall u_1,u_2 \in X. \\ \text{(b)} & \text{There exists } L_A > 0 \text{ such that} \\ & \|Au_1 - Au_2\|_X \leqslant L_A\|u_1 - u_2\|_X \quad \forall u_1,u_2 \in X. \end{cases} \tag{4.3}$$

The nonlinear operator $B : X \to X$ is Lipschitz continuous, i.e., there exists $L_B > 0$ such that

$$\|Bu_1 - Bu_2\|_X \leqslant L_B \|u_1 - u_2\|_X \quad \forall u_1,u_2 \in X. \tag{4.4}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a)} & j(u,\cdot) \text{ is convex and l.s.c. on } X \text{ for all } u \in X. \\ \text{(b)} & \text{There exists } m > 0 \text{ such that} \\ & j(u_1,v_2) - j(u_1,v_1) + j(u_2,v_1) - j(u_2,v_2) \\ & \leqslant m \|u_1 - u_2\|_X \|v_1 - v_2\|_X \quad \forall u_1,u_2,v_1,v_2 \in X. \end{cases} \tag{4.5}$$

Finally, we assume that

$$f \in C([0,T];X) \tag{4.6}$$

and

$$u_0 \in X. \tag{4.7}$$

The following existence, uniqueness and regularity result was proved in Han and Sofonea (2000) and may be found in Han and Sofonea (2002).

Theorem 4.1. Let (4.3)–(4.7), hold. Then:

(1) There exists a unique solution $u \in C^1([0,T];X)$ of problem (4.1) and (4.2).
(2) If u_1 and u_2 are two solutions of (4.1) and (4.2) corresponding to the data $f_1, f_2 \in C([0,T];X)$, then there exists $c > 0$ such that

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_X \leqslant c(\|f_1(t) - f_2(t)\|_X) + \|u_1(t) - u_2(t)\|_X \quad \forall t \in [0,T]. \tag{4.8}$$

(3) If, moreover, $f \in W^{1,p}(0,T;X)$, for some $p \in [1,\infty]$, then the solution satisfies $u \in W^{2,p}(0,T;X)$.

We turn now to the proof of Theorem 3.1. To that end we assume in what follows that (3.7)–(3.18) hold and, everywhere below, we denote by c various positive constants which are independent of time and whose value may change from line to line.

Let $\boldsymbol{\eta} \in C([0,T];Q)$ be given, and in the first step consider the following intermediate mechanical problem in which $\boldsymbol{\eta} = \mathcal{E}^* \nabla \varphi$ is known.

Problem $\mathcal{P}_\eta^1$. Find a displacement field $u_\eta : [0,T] \to V$ such that

$$\begin{cases} (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q + (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q \\ +(\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q + j_{fr}(\boldsymbol{u}_\eta(t), \boldsymbol{v}) - j_{fr}(\boldsymbol{u}_\eta(t), \dot{\boldsymbol{u}}_\eta(t)) \\ \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t))_V \quad \forall \boldsymbol{v} \in V,\ t \in [0,T], \end{cases} \tag{4.9}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{4.10}$$

We have the following result for $\mathcal{P}_\eta^1$.

Lemma 4.2. *There exists $\mu_0 > 0$ depending only on $\Omega, \Gamma_1, \Gamma_3, \mathcal{A}$, and p_v such that, if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, Then:*

(1) There exists a unique solution $u_\eta \in C^1([0,T];V)$ to the problem (4.9) and (4.10).
(2) If u_1 and u_2 are two solutions of (4.9) and (4.10) corresponding to the data $\eta_1,\ \eta_2 \in C([0,T];Q)$, then there exists $c > 0$ such that

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_V \leqslant c(\|\boldsymbol{f}_{\eta_1}(t) - \boldsymbol{f}_{\eta_2}(t)\|_Q + \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V) \quad \forall t \in [0,T]. \tag{4.11}$$

The function $f_\eta : [0,T] \to V$ is defined by

$$(\boldsymbol{f}_\eta(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V - (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \tag{4.12}$$

for all $u,\ v \in V$ and $t \in [0,T]$.

(3) Moreover, if $\eta \in W^{1,p}([0,T];Q)$ for some $p \in [1,\infty]$, then the solution satisfies $u_\eta \in W^{2,p}(0,T;V)$.

Proof of Lemma 4.2. We apply Theorem 4.1 where $X = V$, with the inner product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$. We use the Riesz representation theorem to define the operators $A : V \to V$, $G : V \to V$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \tag{4.13}$$

$$(G\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \tag{4.14}$$

for all $u,\ v \in V$. Assumptions (3.7) and (3.8) imply that the operators A and G satisfy conditions (4.3) and (4.4), respectively.

It follows from (3.6) that the functional j_{fr}, given by (3.19), satisfies condition (4.5)(a). We use again (3.11) and (3.6) to find

$$j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_1) - j_{fr}(\boldsymbol{u}_1, \boldsymbol{v}_2) + j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_{fr}(\boldsymbol{u}_2, \boldsymbol{v}_2) \leqslant \tilde{c}_0^2 L_v \|\mu\|_{L^\infty(\Gamma_3)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V$$

for all $u_1,\ u_2,\ v_1,\ v_2 \in V$, which show that the functional j_{fr} satisfies the condition (4.5)(b) on $X = V$. Moreover, using (3.12) and (3.13) it is easy to see that the function f defined by (3.20) satisfies $f \in W^{1,p}(0,T;V)$ and, keeping in mind that $\eta \in C([0,T];Q)$, we deduce from (4.14) that $f_\eta \in C([0,T];V)$, i.e., f_η satisfies (4.6). Finally, we note that (3.18) shows that condition (4.7) is satisfied, too, and (4.14) shows that if $\eta \in W^{1,p}(0,T;Q)$ then $f_\eta \in W^{1,p}(0,T;V)$. Using now (4.12)–(4.14) we find that Lemma 4.2 is a direct consequence of Theorem 4.1. □

In the next step we use the solution $\boldsymbol{u}_\eta \in C^1([0,T],V)$, obtained in Lemma 4.2, to construct the following variational problem for the electrical potential.

Problem $\mathcal{P}_\eta^2$. Find an electrical potential $\varphi_\eta : [0,T] \to W$ such that

$$(\mathcal{B}\nabla\varphi_\eta(t), \nabla\psi)_W - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \nabla\psi)_W = (\boldsymbol{q}(t), \psi)_W \tag{4.15}$$

for all $\psi \in W,\ t \in [0,T]$.

The well-posedness of problem $\mathcal{P}_\eta^2$ is as follows.

Lemma 4.3. *There exists a unique solution $\varphi_\eta \in W^{1,p}(0,T;W)$ which satisfies (4.15). Moreover, if $\boldsymbol{\varphi}_{\eta_1}$ and φ_{η_2} are the solutions of (4.15) corresponding to $\eta_1,\eta_2 \in C([0,T];Q)$ then, there exists $c > 0$, such that*

$$\|\varphi_{\eta_1}(t) - \boldsymbol{\varphi}_{\eta_2}(t)\|_W \leqslant c\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V \quad \forall t \in [0,T]. \tag{4.16}$$

Proof of Lemma 4.3. Let $t \in [0,T]$. We use the Riesz representation theorem to define the operator $A_\eta(t) : W \to W$ by

$$(A_\eta(t)\varphi, \psi)_W = (\mathcal{B}\nabla\boldsymbol{\varphi}, \nabla\psi)_W - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \nabla\psi)_W \tag{4.17}$$

for all $\varphi,\ \psi \in W$. Let $\varphi_1,\ \varphi_2 \in W$, then assumptions (3.10) imply

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \geqslant m_{\mathcal{B}}\|\varphi_1 - \varphi_2\|_W^2. \tag{4.18}$$

On the other hand, using again (3.9) and (3.10), we have

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \psi)_W \leqslant c_{\mathcal{E}}\|\varphi_1 - \varphi_2\|_W\|\psi\|_W \quad \forall \psi \in W, \tag{4.19}$$

where $c_{\mathcal{E}}$ is a positive constant which depends on $\mathcal{E}$. Thus,

$$\|A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2\|_W \leqslant c_{\mathcal{E}}\|\varphi_1 - \varphi_2\|_W. \tag{4.20}$$

Inequalities (4.18) and (4.19) show that the operator $A_\eta(t)$ is a strongly monotone Lipschitz continuous operator on W and, therefore, there exists a unique element $\varphi_\eta(t) \in W$ such that

$$A_\eta(t)\varphi_\eta(t) = \boldsymbol{q}(t). \tag{4.21}$$

We combine now (4.17) and (4.21) and find that $\varphi_\eta(t) \in W$ is the unique solution of the nonlinear variational Eq. (4.15).

We show next that $\varphi_\eta \in W^{1,p}(0,T;W)$. To this end, let t_1, $t_2 \in [0,T]$ and, for the sake of simplicity, we write $\varphi_\eta(t_i) = \varphi_i$, $u_{\eta\nu}(t_i) = u_i$, $q(t_i) = q_i$, for $i = 1, 2$. Using 4.15, 3.9 and 3.10 we find

$$m_{\mathcal{B}}\|\varphi_1 - \varphi_2\|_W^2 \leqslant c_{\mathcal{E}}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V\|\varphi_1 - \varphi_2\|_W + \|\boldsymbol{q}_1 - \boldsymbol{q}_2\|_W\|\boldsymbol{\varphi}_1 - \varphi_2\|_W, \tag{4.22}$$

where $c_{\mathcal{E}}$ is a positive constant which depends on the piezoelectric tensor $\mathcal{E}$.

Inserting the last inequality in (4.22) yields

$$m_{\mathcal{B}}\|\varphi_1 - \varphi_2\|_W \leqslant c_{\mathcal{E}}\ \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\boldsymbol{q}_1 - \boldsymbol{q}_2\|_W. \tag{4.23}$$

It follows from inequality (4.23) that

$$\|\varphi_1 - \varphi_2\|_W \leqslant c(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\boldsymbol{q}_1 - \boldsymbol{q}_2\|_W). \tag{4.24}$$

We also note that assumptions (3.14) and (3.15), combined with definition (3.21) imply that $q \in W^{1,p}(0,T;W)$. Since $u_\eta \in C^1([0,T];X)$, inequality (4.24) implies that $\varphi_\eta \in W^{1,p}(0,T;W)$.

Let $\eta_1, \eta_2 \in C([0,T];Q)$ and let $\varphi_{\eta_i} = \varphi_i,\ u_{\eta_i} = u_i$, for $i = 1, 2$. We use (4.15) and arguments similar to those used in the proof of (4.23) to obtain

$$m_{\mathcal{B}}\|\varphi_1(t) - \varphi_2(t)\|_W \leqslant c_{\mathcal{E}}\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V$$

for all $t \in [0,T]$. This inequality, leads to (4.16), which concludes the proof. □

We now consider the operator $\Lambda : C([0,T];Q) \to C([0,T];Q)$ defined by

$$\Lambda\boldsymbol{\eta}(t) = \mathcal{E}^*\nabla\varphi_\eta(t) \quad \forall\ \boldsymbol{\eta} \in C([0,T];Q),\ t \in [0,T]. \tag{4.25}$$

We show that Λ has a unique fixed point.

Lemma 4.4. *There exists a unique $\tilde{\boldsymbol{\eta}} \in W^{1,p}(0,T;Q)$ such that $\Lambda\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\eta}}$.*

Proof of Lemma 4.4. Let $\eta_1, \eta_2 \in C([0,T];Q)$ and denote by u_i and φ_i the functions u_{η_i} and φ_{η_i} obtained in Lemmas 4.2 and 4.3, for $i = 1, 2$. Let $t \in [0,T]$. Using (4.25) and (3.9) we obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_Q \leqslant c\ \|\varphi_1(t) - \varphi_2(t)\|_W$$

and, keeping in mind (4.17), we find

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\eta_2(t)\|_Q \leqslant c\ \|u_1(t) - u_2(t)\|_V. \tag{4.26}$$

On the other hand, since $u_i(t) = u_0 + \int_0^t \dot{u}_i(s)\ ds$, we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leqslant \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V ds \tag{4.27}$$

and using this inequality in (4.11) yields

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_V \leqslant c\left(\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q + \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V\ ds\right).$$

It follows now from a Gronwall-type argument that

$$\int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V\ ds \leqslant c\int_0^t \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q ds. \tag{4.28}$$

Combining (4.26)–(4.28) leads to

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_Q \leqslant c\int_0^t \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q ds.$$

Reiterating this inequality n times results in

$$\|\Lambda^n\boldsymbol{\eta}_1(t) - \Lambda^n\boldsymbol{\eta}_2(t)\|_Q \leqslant \frac{c^n}{n!}\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{C([0,T];Q)}.$$

This inequality shows that for a sufficiently large n the operator Λ^n is a contraction on the Banach space $C([0,T];Q)$ and, therefore, there exists a unique element $\tilde{\boldsymbol{\eta}} \in C([0,T];Q)$ such that $\Lambda\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\eta}}$. The regularity $\tilde{\boldsymbol{\eta}} \in W^{1,p}(0,T;Q)$ follows from the fact that $\varphi_{\tilde{\eta}} \in W^{1,p}(0,T;W)$, obtained in Lemma 4.3, combined with the definition (4.25) of the operator Λ. □

We have now all the ingredients to prove the Theorem 3.1 which we complete now.

Existence. Let $\tilde{\boldsymbol{\eta}} \in W^{1,p}(0,T;Q)$ be the fixed point of the operator Λ, and let $\boldsymbol{u}_{\tilde{\eta}}$, $\varphi_{\tilde{\eta}}$ be the solutions of problems $\mathcal{P}^1_\eta$ and $\mathcal{P}^2_\eta$, respectively, for $\boldsymbol{\eta} = \tilde{\boldsymbol{\eta}}$. It follows from (4.25) that $\mathcal{E}^*\nabla\varphi_{\tilde{\eta}} = \tilde{\boldsymbol{\eta}}$ and, therefore, 4.9, 4.10 and 4.15 imply that $(u_{\tilde{\eta}}, \varphi_{\tilde{\eta}})$ is a solution of problem $\mathcal{P}_V$. Properties (3.29) and (3.30) follows from Lemma 4.2 (3) and Lemma 4.3.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ. It can also be obtained by using arguments similar as those used in Rochdi et al. (1998).

5. Conclusion

This paper deals with a mathematical model which describes the quasistatic frictional contact between a piezoelectric body and a deformable foundation. The contact is described with the normal compliance condition and a version of Coulomb's law of friction. The connection between the problem with the Signorini's contact condition and the normal compliance condition that were studied is new and has both theoretical and applied interest. Other results are new, and are reported here for the first time.

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