Journal of the Association of Arab Universities for Basic and Applied Sciences (2015) **18**, 99–101

University of Bahrain

Journal of the Association of Arab Universities for Basic and Applied Sciences

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حلول موجية سولوتونية دورية إضافية لمعادلة (Vakhnenko)

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الملخص:

يختص البحث بتقديم فرضية معدلة من ثلاث حلول موجية لإيجاد حلول سولوتونية دورية ومزدوجة الدورية لمعادلة (Vakhnenko). النتائج تبين أن فرضية الثلاث حلول الموجية يمكن تطبيقها لإيجاد ثلاث أو أثنين من الحلول الموجية لمعادلات تفاضلية جزئية لا خطية ذات أبعاد عليا.

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REVIEW ARTICLE

New periodic solitary wave solutions for an extended generalization of Vakhnenko equation

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Received 27 April 2012; revised 29 April 2014; accepted 4 May 2014
Available online 10 July 2014

KEYWORDS

Extended homoclinic test approach;
Bilinear form;
Extended generalization of Vakhnenko equation;
Periodic solitary wave solutions

Abstract Here, we propose an improvement ansatz in three-wave method, then applying this ansatz to an extended generalization of Vakhnenko equation, we obtain a new periodic type of three-wave solutions including periodic two-solitary solution, doubly periodic solitary solution and breather two-solitary solution, respectively. These results show that the three-wave type of ansatz approach is an effective and simple method for seeking three-wave solutions and two-wave solutions of higher dimensional nonlinear evolution equations.

1. Introduction

The investigation of exact solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. These solutions may give more insight into the physical aspects of the problem modelled by the nonlinear partial differential equations. However, it is not easy to obtain solutions by using analytic methods. So, to solve these nonlinear equations, new methods are needed. In recent years, various methods have been proposed. Recently, a new method, called the Extended three-wave method, is proposed to seek multi-wave solutions of nonlinear partial differential equations. This method was used by some researchers to study various nonlinear partial differential equations in the straightforward way.

In nonlinear science, many important phenomena in various fields can be described by the nonlinear evolution equations. Seeking exact solutions of nonlinear partial differential equations is of great significance as it appears that these (NLPDEs) are mathematical models of complex physics phenomena arising in physics, mechanics, biology, chemistry and engineers. In order to help engineers and physicists to better understand the mechanism that governs these physical models or to better provide knowledge to the physical problem and possible applications, a vast variety of the powerful and direct methods have been derived. Various powerful methods for obtaining explicit travelling solitary wave solutions to nonlinear equations have been proposed (Abdou and Soliman, 2005; Abdou, 2007a,b; Abdou and Zhang, 2009; Abdou, 2008a,b;

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Peer review under responsibility of University of Bahrain.

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http://dx.doi.org/10.1016/j.jaubas.2014.05.001

Abulwafa et al., 2007, 2008; Kim and Sakthivel, 2010; Lee and Sakthivel, 2011; Sakthivel et al., 2010).

One of the most exciting advances of nonlinear science and theoretical physics has been a development of methods to look for exact solutions for nonlinear partial differential equations. Seeking exact solutions of nonlinear equations has been of much interest in recent years because of the availability of symbolic computation Mathematica or Maple. These computer systems allow us to perform some complicated and tedious algebraic and differential calculations on a computer. Multi-wave solutions are important because they reveal the interactions between the inner-waves and the various frequency and velocity components. The whole multi-wave solution, for instance, may sometimes be converted into a single soliton of very high energy that propagates over large regions of space without dispersing and an extremely destructive wave is therefore produced of which the tsunami is a good example. Since all double-wave solutions can be found by using the exp-function method proposed (Abdou and Soliman, 2005; Abdou, 2007a,b; Abdou and Zhang, 2009; Abdou, 2008a,b), we propose an extension of the three-soliton method (Dai et al., 2006, 2008a,b, 2010), namely, three-wave method for finding coupled wave solutions.

This paper deals with an extended generalization of Vakhnenko equation (eGVE) (Li et al., 2010)

$$\frac{\partial}{\partial x}\left[D^2u+\frac{1}{2}pu^2+\beta u\right]+\alpha Du=0, \tag{1}$$

$$D=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}, \tag{2}$$

where α and β are arbitrary non-zero constants. Eq. (1) can be traced to Vakhnenko equation (VE) which was initially presented to model high-frequent waves in a relaxing medium. When $\alpha=1,\ \beta=0$, Eq. (1) is reduced to Vakhnenko equation. When $\alpha=1,\ \beta=0$ is arbitrary non-zero constant, Eq. (1) is reduced to a generalized (eGVE).

The structure of this paper will be organized as follows; In Section 2, with symbolic computation, the bilinear form of Eq. (1) is obtained. In order to illustrate the proposed method, we consider an extended generalization of Vakhnenko equation and new periodic wave solutions are obtained which included periodic two solitary solution, and doubly periodic solitary solution. Finally, conclusion and discussion are given in Section 3.

2. Soliton solutions for an extended generalization of Vakhnenko equation

Making use of the transformation

$$x=T+\int_{-\infty}^{X}U(X',T)dX'+x_0,\quad t=X \tag{3}$$

where $u(x,t)=U(X,T)$ and x_0 is a constant. X and T are two independent variables. Under transformation of (3), Eq. (1) admits to

$$U_{XXT}+\alpha UU_T-\alpha U_X\int_{X}^{-\infty}U(X',T)dX'+\beta U_T+\alpha U_X=0 \tag{4}$$

We introduce a new function W defined by $W(X,T)=\int_{-\infty}^{X}U(X',T)dX'$

$$W_X=U \tag{5}$$

Thus Eq. (4) becomes

$$W_{XXXT}+\alpha[W_XW_{XT}+W_{XX}W_T]+\beta W_{XT}+\alpha W_{XX}=0 \tag{6}$$

According to the Hirota method, we use the transformation

$$W=(lnf)_x \tag{7}$$

Then, Eq. (6) can be expressed as the bilinear equation

$$[D_X^3D_T+\alpha D_X^2+\beta D_XD_T]f\cdot f=0, \tag{8}$$

where $D_x^mD_t^n$ is the Hirota bilinear derivative operator (Hirota, 1980) defined by

$$D_x^mD_t^nf(x,t).g(x,t)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n[f(x,t)g(x',t')]_{x'=x,t'=t}, \tag{9}$$

$$(D_X^3D_T)f\cdot f=2[f_{XXX}f-3f_{XXT}f_X+3f_{XT}f_{XX}-f_{XXX}f_T],$$
$$(D_XD_T)f\cdot f=2[f_{XT}f-f_Xf_T],$$

$$(D_X^2)f\cdot f=2[f_{XX}f-f_Xf_X]. \tag{10}$$

To solve the reduced Eq. (8) by means of the extended homoclinic test function (Dai et al., 2006, 2008a,b, 2010), we suppose a solution of Eq. (8) as follows

$$f(x,t)=e^{-p_1(x-w_1t)}+c_1cos[p_2(x+w_2t)]+c_2e^{p_1(x-w_1t)}, \tag{11}$$

where p_1,p_2,w_1,w_2,c_1 and c_2 are parameters to be determined later.

Substituting Eq. (11) in (8), and equating all coefficients of $[e^{jp_1(x-w_1)t}\,(j=-1,0,1),\ cos(p_2(x+w_2)t,\ sin(p_2(x+w_2)t]$ to zero,we get the set of algebraic equation for $p_i,c_i,w_i\,(i=1,2)$. Solving the set of algebraic equations with the aid of Maple,we have many solutions, in which the following four solutions are chosen

Case(1):

$$p_2=ip_2,\quad p_1=p_1,\quad c_1=c_1,$$
$$w_2=\frac{-2\alpha+4p_1^2w_1+\beta w_1}{4p_1^2w_1+\beta},\quad w_1=w_1,\quad c_2=\frac{c_1^2}{4}. \tag{12}$$

In view of Eq. (12) Eq. (11) admits to

$$f(x,t)=e^{-p_1(x-w_1t)}+c_1cos\left[ip_1\left(x+\left[\frac{-2\alpha+4p_1^2w_1+\beta w_1}{4p_1^2w_1+\beta}\right]t\right)\right]+\frac{c_1^2}{4}e^{p_1(x-w_1t)}. \tag{13}$$

By means of Eqs. (7) and (4), the periodic solitary wave solution of Eq. (11) admits to.

Case(2):

$$p_1=p_1,\quad p_2=\frac{1}{3}\sqrt{3p_1^2+9\beta},\quad c_1=c_1,\quad c_2=0,$$
$$w_2=\frac{3\alpha}{4p_1^2+\beta},\quad w_1=\frac{3\alpha(2p_1^2+3\beta)}{2(4p_1^2+9\beta)p_1^2}. \tag{14}$$

According to Eq. (14) Eq. (11) admits to

$$f(x,t) = e^{-p_1\left(x-\left[\frac{3\alpha(2p_1^2+3\beta)}{2(4p_1^2+9\beta)p_1^2}\right]t\right)} + c_1 cos\left[\frac{1}{3}\sqrt{3p_1^2+9\beta}\left(x+\left[\frac{3\alpha}{4p_1^2+\beta}\right]t\right)\right] \tag{15}$$

Knowing Eq. (15) with Eqs. (7) and (4), the periodic solitary wave solution of Eq. (1) is obtained.

Case(3):

$$p_1 = p_1, \quad p_2 = ip_1, \quad c_1 = c_1, \quad c_2 = \frac{c_1^2}{4},$$
$$w_1 = w_1, \quad w_2 = \frac{-2\alpha + 4p_1^2 + \beta w_1}{4p_1^2 + \beta}. \tag{16}$$

Using Eq. (16) with Eq. (11), we have

$$f(x,t) = e^{-p_1(x-w_1t)} + c_1 cos\left[ip_1\left(x+\left[\frac{-2\alpha+4p_1^2+\beta w_1}{4p_1^2+\beta}\right]t\right)\right] + \frac{c_1^2}{4}e^{p_1(x-w_1t)}. \tag{17}$$

Using Eq. (17) with the aid of Eqs. (7) and (4), the periodic solitary wave solution of Eqs. (1) can be directly evaluated.

Case(4):

$$p_2 = p_2, \quad p_1 = p_1, \quad c_1 = c_1,$$
$$w_2 = \frac{\alpha(p_2^2 + p_1^2 - \beta)}{p_2^4 - 2p_2^2\beta + 2p_1^2p_2^2 + \beta^2 + p_1^4 + 2\beta p_1^2},$$
$$w_1 = \frac{\alpha(p_2^2 + p_1^2 + \beta)}{p_2^4 - 2p_2^2\beta + 2p_1^2p_2^2 + \beta^2 + p_1^4 + 2\beta p_1^2},$$
$$c_2 = -\frac{p_2^2c_1^2(3\beta - 3p_2^2 + p_1^2)}{4(3p_1^2 - p_2^2 + 3\beta)p_1^2} \tag{18}$$

In view of Eq. (18) Eq. (11) admits to

$$f(x,t) = e^{-p_1(x-w_1t)} + c_1 cos[p_2(x + w_2t] + c_2e^{p_1(x-w_1t)}, \tag{19}$$

where w_2, w_1, c_2 are defined by Eq. (18). Using Eq. (19) with Eqs. (7) and (4), the periodic solitary wave solution of Eq. (1) is obtained. For simplicity should be omitte here.

3. Conclusions

In this paper, the proposed a new technique, namely, three-wave approach is used to seek periodic solitary wave solutions for integrable equations, and this method has been used to investigate several equations. The three-wave approach is an extension of the three-soliton method, the main difference is the selection of ansatz, by selecting and substituting a three-wave type of ansatz rather than three-soliton type of ansatz into the bilinear equation, one can effectively obtain exact solutions with three-wave form.

With the aid of bilinear form and the extended homoclinic test approach, we obtain breather-type soliton and two soliton solutions for an extended generalization of Vakhnenko equation, the results show that the extended homoclinic test approach is very effective in finding exact solitary wave solutions for nonlinear evolution equation arising in mathematical physics

It is worthwhile to mention that, the proposed method is reliable and effective and can also be applied to solve other types of higher dimensional integrable and non-integrable systems. Moreover, we investigate different mechanical features of these wave solutions. It is worthnoting that the three-wave approach is effective and simple method for seeking three-wave solutions and two-wave solutions of higher dimensional nonlinear evolution equations.

Acknowledgment

We would like to express our sincere thanks to the referees for their useful comments and discussions.

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