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الاتصال من النوع gm على فراغات التركيبة الدنيا والتوبولوجي المعمم

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الملخص:

في هذا البحث نقدم مفهوم الاتصال من النوع gm والتقارب الى نقطة من النوع GM وذلك على فراغات التركيبة الدنيا والتوبولوجي المعمم. ونقدم ايضا مفهوم الفراغ من النوع $gm-T_2$ والرسم البياني المغلق من النوع gm والرسم البياني المغلق القوي من النوع gm على فراغات التركيبة الدنيا والتوبولوجي المعمم. وقد حصلنا على تمثيلات وخواص متنوعة للدوال المتصلة من النوع gm باستخدام مؤثرات الداخلية والاغلاق المعرفة على كلا من التوبولوجي المعمم g والتركيبة الدنيا m. هذا بالإضافة الى اننا توصلنا الى بعض الخواص المتعلقة بالدوال المتصلة من النوع gm باستخدام مفهوم الفراغ من النوع $gm-T_2$ والرسم البياني المغلق من النوع gm والرسم البياني المغلق القوي من النوع gm.

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ORIGINAL ARTICLE

gm-continuity on generalized topology and minimal structure spaces

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GTMS-space;
gm-continuous function;
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gm-closed graph

Abstract We introduce the concepts of *gm*-continuity, *GM*-converge to a point on generalized topology and minimal structure spaces. Also, we introduce the notions of *gm*-T_2 space, *gm*-closed graph and strongly *gm*-closed graph on generalized topology and minimal structure spaces. We obtain several characterizations and properties of *gm*-continuous functions by using the interior operator and closure operator defined on both a generalized topology g and a minimal structure m. Moreover, we investigate some properties for *gm*-continuous functions by using the notions of *gm*-T_2 space, *gm*-closed graph and strongly *gm*-closed graph.

1. Introduction

The concept of minimal structure (briefly *m*-structure) was introduced by Popa and Noiri (2000). Also they introduced the notions of m_X-open sets and m_X-closed sets and characterize those sets using m_X-closure and m_X-operators, respectively. They introduced the notion of *M*-continuous functions as functions defined between minimal structures. They showed that the *M*-continuous functions on minimal structures have properties similar to those of continuous functions between topological spaces. Császár (2002) introduced the concept of generalized neighborhood systems and generalized topological spaces. He also introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function by using a closure operator defined on generalized neighborhood systems. Recently, the concept of generalized topology and the concept of minimal structure have met the attention of many researchers (see Al-Omari and Noiri, 2013; Boonpok, 2010; Keun and Kim, 2011; Modak, 2013; Noiri and Popa, 2010; Vásquez et al., 2011; Zakari, 2013a,b; Zvina, 2011).

Buadong et al. (2011) introduced the notion of the generalized topology and minimal structure spaces (briefly GTMS). They studied some properties of closed sets on the space. In this paper, we introduce the concept of *gm*-continuous functions on generalized topology and minimal structure spaces. We obtain several characterizations and properties of *gm*-continuous functions by using the interior operator and closure operator defined on both a generalized topology g and a minimal structure m. Moreover, we introduce the notions of *gm*-T_2 space, *gm*-closed graph, strongly *gm*-closed graph and investigate some properties for *gm*-continuous functions by using these notions.

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2. Preliminaries

Definition 2.1 Császár (2002). Let X be a nonempty set and g a collection of subsets of X. Then g is called a *generalized topology* (briefly *GT*) on X if and only if $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $\bigcup_{i\in I} G_i \in g$. We call the pair (X,g) a *generalized topological space* (briefly *GTS*) on X. The elements of g are called *g-open* sets and the complements are called *g-closed sets*.

The closure of a subset A in a generalized topological space (X,g), denoted by $c_g(A)$, is the intersection of generalized closed sets including A, i.e., the smallest g-closed set containing A. The interior of A, denoted by $i_g(A)$, is the union of generalized open sets contained in A, i.e., the largest g-open set contained in A.

Proposition 2.1 Min (2009). *Let (X,g) be a generalized topological space. For subsets A and B of X, the following properties hold.*

(1) $c_g(X-A) = X - i_g(A)$ *and* $i_g(X-A) = X - c_g(A)$;
(2) if $X - A \in g$, *then* $c_g(A) = A$ *and if* $A \in g$, then $i_g(A) = A$;
(3) if $A \subseteq B$, *then* $c_g(A) \subseteq c_g(B)$ *and* $i_g(A) \subseteq i_g(B)$;
(4) $A \subseteq c_g(A)$ *and* $i_g(A) \subseteq A$;
(5) $c_g(c_g(A)) = c_g(A)$ *and* $i_g(i_g(A)) = i_g(A)$.

Proposition 2.2 Min (2009). *Let (X,g) be a generalized topological space and $A \subseteq X$. Then*

(1) $x \in i_g(A)$ *if and only if there exists* $V \in g$ *such that* $x \in V \subseteq A$;
(2) $x \in c_g(A)$ *if and only if* $V \cap A \neq \emptyset$ *for every g*-open set V *containing* x.

Definition 2.2 Császár (2002). Let (X,g_1) and (Y,g_2) be two generalized topological spaces. Then $f : (X,g_1) \longrightarrow (Y,g_2)$ is said to be *g-continuous* if $f^{-1}(V)$ is a g_1-open subset of X for every g_2-open subset of Y.

Definition 2.3 Popa and Noiri (2000). Let X be a nonempty set and $P(X)$ the power set of X. A subfamily m of $P(X)$ is called a *minimal structure* (briefly *m-structure*) on X if $\emptyset \in m$ and $X \in m$.

By (X,m), we denote a nonempty set X with an m-structure m on X and it is called an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*.

Definition 2.4 Popa and Noiri (2000). Let X be a nonempty set and m an m-structure on X. For a subset A of X, the *m-closure* of A denoted by $c_m(A)$ and the *m-interior* of A denoted by $i_m(A)$, are defined as follows:

(1) $c_m(A) = \cap\{F : A \subseteq F, X - F \in m\}$;
(2) $i_m(A) = \cup\{U : U \subseteq A, U \in m\}$.

Lemma 2.1 Maki et al. (1999). *Let X be a nonempty set and m a minimal structure on X. For subsets A and B of X, the following properties hold.*

(1) $c_m(X-A) = X - i_m(A)$ *and* $i_m(X-A) = X - c_m(A)$;
(2) if $X - A \in m$, *then* $c_m(A) = A$ *and if* $A \in m$, *then* $i_m(A) = A$;
(3) $c_m(\emptyset) = \emptyset$, $c_m(X) = X$, $i_m(\emptyset) = \emptyset$ *and* $i_m(X) = X$;
(4) if $A \subseteq B$, *then* $c_m(A) \subseteq c_m(B)$ *and* $i_m(A) \subseteq i_m(B)$;
(5) $A \subseteq c_m(A)$ *and* $i_m(A) \subseteq A$;
(6) $c_m(c_m(A)) = c_m(A)$ *and* $i_m(i_m(A)) = i_m(A)$.

Lemma 2.2 Maki et al. (1999). *Let X be a nonempty set with a minimal structure m and A a subset of X. Then* $x \in c_m(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every m-open set U containing x.*

Set $M(x) = \{U \in m : x \in U\}$Noiri and Popa (2002/2003), we have the following definition.

Definition 2.5 Popa and Noiri (2000). Let (X,m_1) and (Y,m_2) be two minimal structures. Then $f : (X,m_1) \longrightarrow (Y,m_2)$ is said to be *M-continuous* if for $x \in X$ and $V \in M(f(x))$, there is $U \in M(x)$ such that $f(U) \subseteq V$.

Theorem 2.1 Popa and Noiri (2000). *Let* $f : (X,m_1) \longrightarrow (Y,m_2)$ *be a function. Then the following properties are equivalent*:

(1) f is M-continuous;
(2) $f(c_{m_1}(A)) \subseteq c_{m_2}(f(A))$ *for* $A \subseteq X$;
(3) $c_{m_1}(f^{-1}(B)) \subseteq f^{-1}(c_{m_2}(B))$ *for* $B \subseteq Y$;
(4) $f^{-1}(i_{m_2}(B)) \subseteq i_{m_1}(f^{-1}(B))$ *for* $B \subseteq Y$.

Definition 2.6 Buadong et al. (2011). Let X be a nonempty set and let g be a generalized topology and m a minimal structure on X. A triple (X,g,m) is called a *generalized topology and minimal structure space* (briefly *GTMS space*).

Definition 2.7 Buadong et al. (2011). Let (X,g,m) be a GTMS space. A subset A of X is said to be a *gm-closed* if $c_g(c_m(A)) = A$. A subset A of X is said to be a *mg-closed* if $c_m(c_g(A)) = A$.

Lemma 2.3 Buadong et al. (2011). *Let (X,g,m) be a GTMS space and $A \subseteq X$. Then*

(1) A is gm-closed if and only if $c_m(A) = A$ *and* $c_g(A) = A$;
(2) A is mg-closed if and only if $c_m(A) = A$ *and* $c_g(A) = A$.

Proposition 2.3 Buadong et al. (2011)*Let (X,g,m) be a GTMS space and $A \subseteq X$. Then A is gm-closed if and only if A is mg-closed.*

Definition 2.8 Buadong et al. (2011) Let (X,g,m) be a GTMS space and A a subset of X. Then A is said to be *closed* if A is *gm*-closed. The complement of a closed set is said to be an *open set*.

Through this paper, $GMO(X)$ denotes to the collection of all open subsets of a GTMS space (X,g,m).

Definition 2.9 Buadong et al. (2011). Let (X,g,m) be a GTMS space. A subset A of X is said to be a *s-closed* if $c_g(A) = c_m(A)$. A subset A of X is said to be a *c-closed* if $c_g(c_m(A)) = c_m(c_g(A))$. The complement of a *s*-closed (resp. *c*-closed) set is called a *s-open* (resp. *c-open*) *set*.

Proposition 2.4 Buadong et al. (2011). *Let* (X, g, m) *be a GTMS space and* $A \subseteq X$. *If A is closed, then A is s-closed.*

Proposition 2.5 Buadong et al. (2011). *Let* (X, g, m) *be a GTMS space and* $A \subseteq X$. *If A is s-closed, then A is c-closed.*

Theorem 2.2 Buadong et al. (2011). *Let* (X, g, m) *be a GTMS space and* $A \subseteq X$. *Then*

(1) A is closed if and only if there exists a s-closed set B such that $A = c_g(B)$;
(2) A is closed if and only if there exists a c-closed set B such that $A = c_g(c_m(B))$.

3. *gm*-continuous functions

Definition 3.1 A function $f: (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ is said to be *gm-continuous* at a point $x \in X$ if for every open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq V$.

A function $f: (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ is said to be *gm-continuous* if it has this property at each point $x \in X$.

Theorem 3.1 *Let* $f: (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ *be a function. Then the following properties are equivalent*:

(1) f is gm-continuous at $x \in X$;
(2) $x \in i_{g_1}(i_{m_1}(f^{-1}(V)))$ *for every open subset V of Y containing* $f(x)$;
(3) $x \in i_{g_1}(i_{m_1}(f^{-1}(B)))$ *for every c*-open subset B of Y *such that* $x \in f^{-1}(i_{g_2}(i_{m_2}(B)))$;
(4) $x \in i_{g_1}(i_{m_1}(f^{-1}(B)))$ *for every s*-open subset B of Y *such that* $x \in f^{-1}(i_{g_2}(i_{m_2}(B)))$;
(5) $x \in f^{-1}(F)$ *for every closed subset F of Y such that* $x \in c_{g_1}(c_{m_1}(f^{-1}(F)))$.
(6) $x \in f^{-1}(c_{g_2}(c_{m_2}(B)))$ *for every c-closed subset B of Y such that* $x \in c_{g_1}(c_{m_1}(f^{-1}(B)))$;
(7) $x \in f^{-1}(c_{g_2}(c_{m_2}(B)))$ *for every s-closed subset B of Y such that* $x \in c_{g_1}(c_{m_1}(f^{-1}(B)))$.

Proof

(1) ⇒ (2) Let V be any open subset of Y containing $f(x)$. By (1), there exists an open subset U of X containing x such that $f(U) \subseteq V$. Since U is an open subset of X, we have $x \in i_{g_1}(i_{m_1}(f^{-1}(V)))$.

(2) ⇒ (3) Let B be any c-open subset of Y such that $x \in f^{-1}(i_{g_2}(i_{m_2}(B)))$. Then $f(x) \in i_{g_2}(i_{m_2}(B))$. Since B is c-open, then $i_{g_2}(i_{m_2}(B))$ is open, from Theorem 2.2 (2). By (2), we have $x \in i_{g_1}(i_{m_1}(f^{-1}(i_{g_2}(i_{m_2}(B))))) \subseteq i_{g_1}(i_{m_1}(f^{-1}(B)))$. Hence $x \in i_{g_1}(i_{m_1}(f^{-1}(B)))$.

(3) ⇒ (4) Let B be any s-open subset of Y such that $x \in f^{-1}(i_{g_2}(i_{m_2}(B)))$. Then B is c-open, from Proposition 2.5. By (3), we get $x \in i_{g_1}(i_{m_1}(f^{-1}(B)))$.

(4) ⇒ (5) Let F be any closed subset of Y such that $x \notin f^{-1}(F)$. Then $x \in X - f^{-1}(F) = f^{-1}(Y - F)$. By (4), $x \in i_{g_1}(i_{m_1}(f^{-1}(Y - F))) = i_{g_1}(X - c_{m_1}(f^{-1}(F))) = X - c_{g_1}(c_{m_1}(f^{-1}(F)))$, Since $Y - F$ is open subset of X and then s-open, from Proposition 2.4. Hence we have $x \notin c_{g_1}(c_{m_1}(f^{-1}(F)))$.

(5) ⇒ (6) Let B be any c-closed subset of Y such that $x \in c_{g_1}(c_{m_1}(f^{-1}(B)))$. Then $x \in c_{g_1}(c_{m_1}(f^{-1}(c_{g_2}(c_{m_2}(B)))))$. Since B is c-closed, then $c_{g_2}(c_{m_2}(B))$ is closed, from Theorem 2.2 (2). By (5), $x \in f^{-1}(c_{g_2}(c_{m_2}(B)))$.

(6) ⇒ (7) Obvious.

(7) ⇒ (2) Let $x \in X$ and V be an open subset of Y containing $f(x)$. Then $x \notin f^{-1}(Y - V) = f^{-1}(c_{g_2}(c_{m_2}(Y - V)))$, since $Y - V$ is closed. By (7), $x \notin c_{g_1}(c_{m_1}(f^{-1}(Y - V)))$. Hence $x \in i_{g_1}(i_{m_1}(f^{-1}(V)))$.

(2) ⇒ (1) Let V be any open subset of Y containing $f(x)$. Then by (2), $x \in i_{g_1}(i_{m_1}(f^{-1}(V)))$, so there exists an open set $U = i_{g_1}(i_{m_1}(f^{-1}(V)))$ containing x such that $f(U) \subseteq V$. Hence, f is gm-continuous at $x \in X$. □

Theorem 3.2. *Let* $f: (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ *be a function. Then the following properties are equivalent*:

(1) f is gm-continuous;
(2) $f^{-1}(V)$ *is an open subset of X for every open subset V of Y*;
(3) $f^{-1}(i_{g_2}(i_{m_2}(B))) \subseteq i_{g_1}(i_{m_1}(f^{-1}(B)))$ for every c-open subset B of Y;
(4) $f^{-1}(i_{g_2}(i_{m_2}(B))) \subseteq i_{g_1}(i_{m_1}(f^{-1}(B)))$ for every s-open subset B of Y;
(5) $f^{-1}(F)$ *is a closed subset of X for every closed subset F of Y*;
(6) $c_{g_1}(c_{m_1}(f^{-1}(B))) \subseteq f^{-1}(c_{g_2}(c_{m_2}(B)))$ *for every c-closed subset B of Y*;
(7) $c_{g_1}(c_{m_1}(f^{-1}(B))) \subseteq f^{-1}(c_{g_2}(c_{m_2}(B)))$ *for every s-closed subset B of Y*.

Proof

(1) ⇒ (2) Let V be any open subset of Y such that $x \in f^{-1}(V)$. Then $f(x) \in V$. By (1), there exists an open subset U of X containing x such that $f(U) \subseteq V$. Since U is an open subset of X, we have $x \in i_{g_1}(i_{m_1}(f^{-1}(V)))$. Thus $f^{-1}(V) \subseteq i_{g_1}(i_{m_1}(f^{-1}(V)))$. Consequently, $f^{-1}(V) = i_{g_1}(i_{m_1}(f^{-1}(V)))$ and then $f^{-1}(V)$ is open.

(2) ⇒ (3) Let B be any c-open subset of Y such that $x \in f^{-1}(i_{g_2}(i_{m_2}(B)))$. Then $f(x) \in i_{g_2}(i_{m_2}(B))$. By (2), $f^{-1}(i_{g_2}(i_{m_2}(B)))$ is open. Then $x \in i_{g_1}(i_{m_1}(f^{-1}(i_{g_2}(i_{m_2}(B))))) \subseteq i_{g_1}(i_{m_1}(f^{-1}(B)))$. Thus $x \in i_{g_1}(i_{m_1}(f^{-1}(B)))$. Hence $f^{-1}(i_{g_2}(i_{m_2}(B))) \subseteq i_{g_1}(i_{m_1}(f^{-1}(B)))$.

(3) ⇒ (4) Obvious.

(4) ⇒ (5) Let F be any closed subset of Y such that $x \notin f^{-1}(F)$. Then $x \in X - f^{-1}(F) = f^{-1}(Y - F) = f^{-1}(i_{g_2}(i_{m_2}(Y - F)))$, since $Y - F$ is open. By (4), $x \in i_{g_1}(i_{m_1}(f^{-1}(Y - F))) = i_{g_1}(X - c_{m_1}(f^{-1}(F))) = X - c_{g_1}(c_{m_1}(f^{-1}(F)))$. Then we have $x \notin c_{g_1}(c_{m_1}(f^{-1}(F)))$. Hence $f^{-1}(F) = c_{g_1}(c_{m_1}(f^{-1}(F)))$ and then F is closed.

(5) ⇒ (6) Let B be any c-closed subset of Y such that $x \in c_{g_1}(c_{m_1}(f^{-1}(B)))$. Then $x \in c_{g_1}(c_{m_1}(f^{-1}(c_{g_2}(c_{m_2}(B)))))$. Since $c_{g_2}(c_{m_2}(B))$ is closed, then by (5), we get $f^{-1}(c_{g_2}(c_{m_2}(B)))$ is closed and hence $c_{g_1}(c_{m_1}(f^{-1}(B))) \subseteq f^{-1}(c_{g_2}(c_{m_2}(B)))$.

(6) ⇒ (7) Obvious.

(7) ⇒ (1) Let F be any closed subset of Y and $x \in c_{g_1}(c_{m_1}(f^{-1}(F)))$. By (7), we have $x \in f^{-1}(F)$. By Theorem 3.1(5), f is gm-continuous. □

Theorem 3.3 *Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be a function. Then the following properties are equivalent*:

(1) f is gm-continuous;
(2) there exists a s-open subset U of X such that $f^{-1}(V) = i_{g_1}(U)$ for every open subset V of Y;
(3) there exists a c-open subset U of X such that $f^{-1}(V) = i_{g_1}(i_{m_1}(U))$ for every open subset V of Y;
(4) there exists a c-closed subset H of X such that $f^{-1}(F) = c_{g_1}(c_{m_1}(H))$ for every closed subset F of Y;
(5) there exists a s-closed subset H of X such that $f^{-1}(F) = c_{g_1}(H)$ for every closed subset F of Y.

Proof

(1) ⇒ (2) Let V be any open subset of Y. From Theorem 3.2(2), $f^{-1}(V)$ is an open subset of X, since f is gm-continuous. Then, there exists a s-open subset U of X such that $f^{-1}(V) = i_{g_1}(U)$, from Theorem 2.2(1).

(12) ⇒ (3) Let V be any open subset of Y. By (2), there exists a s-open subset U of X such that $f^{-1}(V) = i_{g_1}(U)$. Since U is c-open, then $f^{-1}(V) = i_{g_1}(i_{m_1}(U))$.

(3) ⇒ (4) Let F be a closed subset of Y. By (3), there exists a c-open subset U of X such that $f^{-1}(X - F) = i_{g_1}(i_{m_1}(U))$, since $X - F$ is open. Hence, $X - f^{-1}(F) = i_{g_1}(i_{m_1}(U))$. Thus, $f^{-1}(F) = c_{g_1}(c_{m_1}(X - U))$. If $H = X - U$, then H is a c-closed subset of X such that $f^{-1}(F) = c_{g_1}(c_{m_1}(H))$.

(4) ⇒ (1) Let F be a closed subset of Y. By (4), there exists a c-closed subset H of X such that $f^{-1}(F) = c_{g_1}(c_{m_1}(H))$. By Theorem 2.2 (2), $f^{-1}(F)$ is a closed subset of X. Thus, f is gm-continuous, from Theorem 3.2 (5).

(1) ⇒ (5) Follows directly from Theorem 3.2 (5) and Theorem 2.2 (1). □

Definition 3.2 A GTMS-space (X, g, m) is called a *s-discrete GTMS-space* if every subset of X is s-closed.

Theorem 3.4 *Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be a function such that (Y, g_2, m_2) be a s-discrete GTMS-space. Then the following properties are equivalent*:

(1) f is gm-continuous;
(2) $f(c_{g_1}(c_{m_1}(A))) \subseteq c_{g_2}(c_{m_2}(f(A)))$ for every subset A of X;
(3) $i_{g_2}(i_{m_2}(f(A)) \subseteq f(i_{g_1}(i_{m_1}(A))))$ for every subset A of X.

Proof

(1) ⇒ (2) Suppose that A is a subset of X. Since $f(A) \subseteq c_{g_2}(c_{m_2}(f(A)))$, then $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(c_{g_2}(c_{m_2}(f(A))))$. By Theorem 2.2(1) , $c_{g_2}(c_{m_2}(f(A)))$ is closed, since $f(A)$ is s-closed subset of Y. Thus, $f^{-1}(c_{g_2}(c_{m_2}(f(A))))$ is closed subset of X, since f is gm-continuous. Hence, $c_{g_1}(c_{m_1}(A)) \subseteq f^{-1}(c_{g_2}(c_{m_2}(f(A))))$, this implies that $f(c_{g_1}(c_{m_1}(A))) \subseteq c_{g_2}(c_{m_2}(f(A)))$.

(2) ⇒ (3) Obvious.

(2) ⇒ (1) Let F be a closed subset of Y. Then $c_{g_2}(c_{m_2}(F)) = F$. By (2), we have $c_g(c_m(f^{-1}(F))) \subseteq f^{-1}(F)$. Thus, $f^{-1}(F)$ is closed subset of X and hence f is gm-continuous. □

Theorem 3.5. *Let $f : (X, g_1) \longrightarrow (Y, g_2)$ be a g-continuous function and $f : (X, m_1) \longrightarrow (Y, m_2)$ be a M-continuous function. Then the function $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ is gm-continuous.*

Proof. Let V be any open subset of Y. Since f is g-continuous, we have $f^{-1}(V) = i_{g_1}(f^{-1}(V))$. By Theorem 2.1, we have $f^{-1}(V) = f^{-1}(i_{m_2}(V)) \subseteq i_{m_1}(f^{-1}(V))$, since f is M-continuous. Thus $f^{-1}(V) \subseteq i_{m_1}(i_{g_1}(f^{-1}(V)))$. Consequently, $f^{-1}(V)$ is open subset of Y. □

From the above theorem, we have the following implication but the reverse relation may not be true in general.

$M -$ continuity $+ g -$ continuity $\Rightarrow gm -$ continuity

Example 3.1. Let $X = Y = \{a, b, c\}$, $g_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $g_2 = \{\emptyset, \{a\}, \{a, b\}\}$, $m_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $m_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. If $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ defined by $f(a) = a, f(b) = b, f(c) = c$, then f is gm-continuous but it is not M-continuous, since $\{a, c\} \subseteq Y$ but $f^{-1}(i_{m_2}(\{a, c\})) = \{a, c\} \nsubseteq i_{m_1}(f^{-1}(\{a, c\})) = \{a\}$.

Example 3.2. Let $X = Y = \{a, b, c\}$, $g_1 = \{\emptyset, \{c\}\{a, c\}\}$, $g_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$, $m_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$, $m_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. If $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ defined by $f(a) = c$, $f(b) = b, f(c) = a$, then f is gm-continuous but it is not g-continuous, since $\{c\}$ is g_2-open but $f^{-1}(\{c\}) = \{a\}$ is not g_1-open .

Definition 3.3. Let X be a nonempty set. Then the collection Γ of subsets of X is called a *gm-family* on X if $\cap\Gamma \neq \emptyset$.

Definition 3.4. Let (X, g, m) be a GTMS space and Γ a gm-family on X. Then we say that a gm-family Γ*GM-converges* to $x \in X$ if $GM(x) \subseteq \Gamma$, where $GM(x) = \{U \in GMO(X) : x \in U\}$.

Let $f : (X, g_1) \longrightarrow (Y, g_2)$ be a function. Then it is obvious that $f(\Gamma) = \{f(A) : A \in \Gamma\}$ is a gm-family on Y.

Lemma 3.1. *Let (X, g, m) be a GTMS space. Then the following properties hold.*

(1) Let U be an open subset of X. Then $x \in U$ if and only if $U \in \Gamma$ whenever a gm-family Γ GM-converges to x;
(2) Let F be a closed subset of X. Then $x \in F$ if and only if there exists a gm-family Γ such that Γ GM-converges to x and $X - F \notin \Gamma$.

Proof. (1) Let U be an open subset of X and $x \in X$. Then $U \in GM(x)$. Hence if a *gm*-family Γ converges to x, then it follows that $U \in \Gamma$ since $GM(x) \subseteq \Gamma$.

Conversely, suppose that for every *gm*-family Γ converges to $x, U \in \Gamma$. Then since $GM(x) GM$-converges to x, by hypothesis, we get $U \in GM(x)$ and hence $x \in U$.

(2) Let F be a closed subset of X and $x \in F$. Then $X - F \notin GM(x)$. Let $\Gamma = GM(x)$, then Γ is a *gm*-family satisfying the condition.

For the converse, Let Γ be a *gm*-family GM-converges to x and $X - F \notin \Gamma$. Since $GM(x) \subseteq \Gamma$, we get $X - F \notin GM(x)$ and so $x \in F$. □

Theorem 3.6. *Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be a bijective function. Then f is gm-continuous if and only if for a gm-family Γ GM-converging to $x \in X$, $f(\Gamma)$ GM-converges to $f(x)$.*

Proof. Suppose f is *gm*-continuous and Γ is a *gm*-family GM-converging to $x \in X$. Let $V \in GM(f(x))$. Then $x \in f^{-1}(V)$. By *gm*-continuity, $f^{-1}(V)$ is an open subset of X. By Lemma 3.1(1), we have $f^{-1}(V) \in \Gamma$. By surjectivity, $V \in f(\Gamma)$ and hence $GM(f(x)) \subseteq f(\Gamma)$. Consequently, $f(\Gamma) GM$-converges to $f(x)$.

For the converse, Let $x \in X$ and V be any open subset of Y containing $f(x)$. Then $V \in GM(f(x))$. Since $GM(x)$ GM-convergences to x, by hypothesis, we get $GM(f(x)) \subseteq f(GM(x))$. From F is injectivity, it follows $f^{-1}(V) \in GM(x)$. If $U = f^{-1}(V)$, then U is an open subset of X containing x such that $f(U) \subseteq V$. Hence f is *gm*-continuous. □

The *gm*-continuity does not satisfy if the condition of Theorem 3.6 doesn't exist and vice versa.

Example 3.3. Let $X = Y = \{a, b, c\}$, $g_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $g_2 = \{\emptyset, \{a\}, \{a, b\}\}$, $m_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$, $m_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. If $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ defined by $f(a) = c, f(b) = a, f(c) = b$. Then f is bijective function and we have, $GM(a) GM$-converges to a, but $f(GM(a))$ does not converge to $f(a)$, since $f(GM(a)) = \{\{c\}, \{a, c\}\} \nsubseteq GM(f(a)) = \emptyset$. On the other hand, $\{a, b\}$ is an open subset of Y but $f^{-1}(\{a, b\}) = \{b, c\}$ is not open in X. Thus, f is not *gm*-continuous.

Definition 3.5. Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be a function. Then f has a *gm-closed graph (resp. strongly gm-closed graph)* if for each $(x, y) \in (X \times Y) - G(f)$, there exists an open subset U containing x and an open subset V containing y such that $(U \times V) \cap G(f) = \emptyset$ (resp. $(U \times c_{g_2}(c_{m_2}(V))) \cap G(f) = \emptyset$).

Lemma 3.2. *Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be a function. Then f has a gm-closed graph (resp. strongly gm-closed graph) if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exists an open subset U containing x and an open subset V containing y such that $f(U) \cap V = \emptyset$ (resp. $f(U) \cap c_{g_2}(c_{m_2}(V))) = \emptyset$).*

Proof. Obvious. □

Definition 3.6. A GTMS space (X, g, m) is called *gm-T_2 space* if for any pair of distinct points x and y in X, there are disjoint open sets U, V such that $x \in U, y \in V$.

Theorem 3.7. *Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be a gm-continuous function. If (Y, g_2, m_2) is gm-T_2 space, then f has a strongly gm-closed graph.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $f(x) \neq y$. Since Y is *gm*-T_2 space, there are disjoint open sets U, V such that $f(x) \in U, y \in V$. This implies that $c_{g_2}(c_{m_2}(V))) \cap U = \emptyset$. For $f(x) \in U$, from *gm*-continuity of f, there exists an open set G containing x such that $f(G) \subseteq U$. Consequently, we can say that there exist open sets G, V containing x, y respectively, such that $f(G) \cap c_{g_2}(c_{m_2}(V))) = \emptyset$ and so by Lemma 3.2, f has a strong *gm*-closed graph. □

Corollary 3.8. *Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be a gm-continuous function. If (Y, g_2, m_2) is gm-T_2 space, then f has gm-closed graph.*

Theorem 3.9. *Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be an injective gm-continuous function and Y is gm-T_2 space. Then X is gm-T_2 space.*

Proof. Obvious. □

Theorem 3.10. *Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be an injective gm-continuous function with gm-closed graph. Then X is gm-T_2 space.*

Proof. Let x_1 and x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since f has *gm*-closed graph, there exists an open subset U containing x_1 and an open subset V containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Since f is *gm*-continuous, $f^{-1}(V)$ is an open subset containing x_2 such that $U \cap f^{-1}(V) = \emptyset$. Hence X is *gm*-T_2 space. □

Corollary 3.11. *Let $f : (X, g_1, m_1) \longrightarrow (Y, g_2, m_2)$ be an injective gm-continuous function with strongly gm-closed graph. Then X is gm-T_2 space.*

4. Conclusions

Through this paper, we introduced many concepts on generalized topology and minimal structure spaces namely, *gm*-continuous functions, *gm*-T_2 space, *gm*-closed graph and strongly *gm*-closed graph. We obtained several characterizations and properties of *gm*-continuous functions by using these notions. It is worth mentioning that for the first time, we offered the concept of *gm*-continuity on generalized topology and minimal structure spaces to open the horizons for researchers to expand the study of other types of continuity on these spaces. Although, we could not introduce the concept of the continuity on generalized topology and minimal structure spaces as previously defined by Duangphui et al. (2011), we were able to provide a appropriate definition which enables us to find the relationship between it and the continuity on generalized topological spaces (Császár, 2002) and the continuity on minimal structures (Popa and Noiri, 2000), as we have seen in Theorem 3.5. The difficulty is due to the generalized topology and minimal structure spaces are determined by two different types of spaces (generalized topological space and minimal structure) in reverse to the bigeneralized topological spaces, which are determined by the same type of spaces.

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