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حل مسألة الشروط الحدية ذات الثلاث نقاط من الرتبة الرابعة

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الملخص:

يقترح في هذا البحث طريقة حسابية لحل معادلة تفاضلية (خطية، لاخطية) من الرتبة الرابعة بشروط حدية عند ثلاث نقاط، وكذلك محل مجموعة من مسائل الشروط الحدية اللاخطية ذات الثلاث نقاط. تعتمد الطريقة على استخدام مفكوك أدوميان (ADM)، طريقة إعادة استنتاج الحل الرئيسي (RKM). الحلول العددية للحلول التقريبية، والذي تأخذ شكل متسلسلات وتؤكد دقة الطريقة المقترحة.

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ORIGINAL ARTICLE

Solution of fourth order three-point boundary value problem using ADM and RKM

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Abstract In this paper, a computational method is proposed, for solving linear and nonlinear fourth order three-point boundary value problem (BVP) and the system of nonlinear BVP. This method is based on the Adomian decomposition method (ADM) and the reproducing kernel method (RKM). The solution of linear fourth order three-point boundary value problem (BVP) is determined by the reproducing kernel method, and the solution of nonlinear fourth order three-point BVP is determined using the combination of Adomian decomposition method and reproducing kernel method. The approximate solutions are given in the form of series. Numerical results are shown to illustrate the accuracy of the present method.

1. Introduction

Fourth order ordinary differential equations are models for bending or deformation of elastic beams and therefore have important applications in engineering and physical sciences. Two-point and multi-point boundary value problems for fourth order ordinary differential equations have attracted a lot of attention. Two-point boundary value problems have been extensively studied. Multi-point boundary value problems arise in a variety of Applied Mathematics and Physics. Many authors have studied the beam equation under various boundary conditions and by different approaches (Graef et al., 2003, 2009). Sufficient conditions for the existence and non-existence of positive solutions for three-point boundary value problems are established by Graef et al. (2009). Siddiqi and Ghazala (2007) determined the solution of a system of fourth order boundary value problems using cubic non-polynomial spline method. Ghazala and Hamood (2012) used RKM for the solution of fourth order singularly perturbed boundary value problem. Ghazala and Hamood (2011) used RKM for the solution of fifth order boundary value problem.

Adomian decomposition method has been used to solve linear and nonlinear ordinary differential equations (Abbaoui and Cherruault, 1995; Biazar and Shafiof, 2007; Mestrovic, 2007). This method provides the solution in a rapid convergent series with computable terms. However, for the solution of boundary value problems using ADM, it is necessary to determine some unknown parameters and therefore, it is required to solve nonlinear algebraic equations. Geng and Cui (2011) proposed a method for solving nonlinear second order two-point BVP by the combination of ADM and RKM. The fourth order three-point BVP described in this paper has been solved by extending the method developed by Geng and Cui (2011).

The fourth order beam equation can be considered, as

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$$\begin{cases} a_0(x)u^{(4)}(x)+a_1(x)u^{(3)}(x)+a_2(x)u^{(2)}(x)+a_3(x)u^{(1)}(x) \\ +a_4(x)u(x)=f(x)+g(x,u(x)),\ 0\leqslant x\leqslant 1 \\ u(0)=\gamma_0,\quad u^{(1)}(0)=\gamma_1,\quad u(\alpha)=\gamma_2,\quad u(1)=\gamma_3. \end{cases} \tag{1.1}$$

where $\alpha\in[0,1]$ is a constant, $a_i(x),f(x)\in C[0,1],i=0,1,2,3,4$ and $g(x,u(x))$ is infinitely differentiable w.r.t $u(x)$.

The nonhomogeneous boundary conditions

$$u(0)=\gamma_0,\quad u^{(1)}(0)=\gamma_1,\quad u(\alpha)=\gamma_2,\quad u(1)=\gamma_3$$

can be reduced to homogeneous boundary conditions, as

$$u(0)=0,\quad u^{(1)}(0)=0,\quad u(\alpha)=0,\quad u(1)=0.$$

The operator form of Eq. (1.1) can be considered, as

$$\begin{cases} Lu(x)=f(x)+g(x,u(x)),\ 0\leqslant x\leqslant 1 \\ u(0)=\gamma_0,\quad u^{(1)}(0)=\gamma_1,\quad u(\alpha)=\gamma_2,\quad u(1)=\gamma_3, \end{cases} \tag{1.2}$$

where $L=a_0(x)\frac{d^{(4)}}{dx}+a_1(x)\frac{d^3}{dx}+a_2(x)\frac{d^2}{dx}+a_3(x)\frac{d}{dx}+a_4(x)$.

The rest of the paper is organized as under:

In Section 2, definition and a derivation of reproducing kernel spaces are presented. In Section 3, a reproducing kernel satisfying three-point boundary conditions is constructed and the solution for linear fourth order BVP using reproducing kernel Hilbert space is presented. In Section 4, Adomian decomposition method is discussed and then combined formula of ADM and RKM is determined for the solution of nonlinear BVP. In Section 5, four examples are presented to demonstrate the usefulness of the method.

2. Reproducing kernel spaces

Definition 2.1. Let H be a Hilbert space of functions on a set X. Let $\langle f,g\rangle$ be the inner product and $\|f\|=\sqrt{\langle f,f\rangle}$ be the norm in H, for f and $g\in H$. The complex valued function $K(x,y)$ of x and y in X is called a reproducing kernel of H if the following are satisfied:

(i) For every x, $K_x(y)=K(x,y)$ as a function of y belongs to H.
(ii) The reproducing property: for every $x\in X$ and every $f\in H, f(x)=\langle f,K_x\rangle$.

2.1. Reproducing kernel space $W_2^5[0,1]$

The reproducing kernel space $W_2^5[0,1]$ is defined by

$W_2^5[0,1]=\{u(x)|u^{(i)}(x),i=0,1,2,3,4$ are absolutely continuous real valued functions in $[0,1],u^{(5)}(x)\in L^2[0,1]$, $u(0)=0,u^{(1)}(0)=0,u(1)=0\}$.

The inner product and norm in $W_2^5[0,1]$ are defined respectively, as

$$\langle u(x),v(x)\rangle=\sum_{i=0}^{4}u^{(i)}(0)v^{(i)}(0)+\int_0^1 u^{(5)}(x)v^{(5)}(x)dx,$$

$$u(x),v(x)\in W_2^5[0,1]. \tag{2.1}$$

$$\|u\|=\sqrt{\langle u(x),u(x)\rangle},\quad u(x)\in W_2^5[0,1]. \tag{2.2}$$

2.2. Reproducing kernel space $W_2^1[0,1]$

The reproducing kernel space $W_2^1[0,1]$ is defined by

$W_2^1[0,1]=\{u(x)|u(x)$ is absolutely continuous real valued function in $[0,1],u^{(1)}(x)\in L^2[0,1]\}$.

Also the inner product and norm are defined respectively, as

$$\langle u(x),v(x)\rangle=u(0)v(0)+\int_0^1 u^{(1)}(x)v^{(1)}(x)dx,$$

$$u(x),v(x)\in W_2^1[0,1]. \tag{2.3}$$

$$\|u\|=\sqrt{\langle u(x),u(x)\rangle},\quad u(x)\in W_2^1[0,1]. \tag{2.4}$$

$W_2^1[0,1]$ is reproducing kernel Hilbert space, and its reproducing kernel is

$$\overline{R}_x(y)=\begin{cases}1+x, & x\leqslant y,\\ 1+y, & x>y.\end{cases}$$

Theorem 2.1. *The space $W_2^5[0,1]$ is a reproducing kernel Hilbert space i.e. for all $u(y)\in W_2^5[0,1]$ and each fixed $x,y\in[0,1]$, there exists $R_x(y)\in W_2^5[0,1]$ such that $\langle u(y),R_x(y)\rangle=u(x)$, and $R_x(y)$ is called the reproducing kernel function of space $W_2^5[0,1]$. The reproducing kernel function $R_x(y)$ is given by*

$$R_x(y)=\begin{cases}k(x,y)=\sum_{i=0}^{9}a_iy^i, & y\leqslant x,\\ k(y,x)=\sum_{i=0}^{9}b_iy^i, & y>x,\end{cases}$$

where

$$\begin{cases} k(x,y)=\frac{1}{101500}[y^2(-90720(-1+x)x^2(10780 \\ \quad +x(700+x(70+x(-56+x(28+(-8+x)x))))) \\ \quad -10080(-1+x)x^2(-90720 \\ \quad +x(700+x(70+x(-56+x(28+(-8+x)x)))))y \\ \quad -630(-1+x)x^2(-90720+x(-100800 \\ \quad +x(70+x(-56+x(28+(-8+x)x)))))y^2 \\ \quad -126(-1+x)x^2(-90720+x(-100800 \\ \quad +x(70+x(-56+x(28+(-8+x)x)))))y^3 \\ \quad +84(-1+x)x^2(-90720+x(700+x(70+x(-56 \\ \quad +x(28+(-8+x)x)))))y^4 \\ \quad -36(-1+x)x^2(10780+x(700+x(70+x(-56 \\ \quad +x(28+(-8+x)x)))))y^5+9(-1+x)x(101500 \\ \quad +x(10780+x(700+x(70 \\ \quad +x(-56+x(28+(-8+x)x))))))y^6 \\ \quad -(-101500+x^2(90720+x(10080+x(630 \\ \quad +x(126+x(-84+x(36+(-9+x)x)))))))y^7)]. \end{cases}$$

Proof. Since $R_x(y)\in W_2^5[0,1]$, so

$$R_x(0)=0,\quad R_x^{(1)}(0)=0,\quad R_x(1)=0, \tag{2.5}$$

using the inner product of space $W_2^5[0,1]$

$$\langle u(y),R_x(y)\rangle=\sum_{i=0}^{4}u^{(i)}(0)R_x^{(i)}(0)+\int_0^1 u^{(5)}(y)R_x^{(5)}(y)dy. \tag{2.6}$$

Integrating Eq. (2.6), gives

$$\begin{aligned}\langle u(y), R_x(y)\rangle = {} & u(0)R_x(0) + u^{(1)}(0)R_x^{(1)}(0) + u^{(2)}(0)R_x^{(2)}(0) \\ & + u^{(3)}(0)R_x^{(3)}(0) + u^{(4)}(0)R_x^{(4)}(0) \\ & + u^{(4)}(y)R_x^{(5)}(y)|_0^1 - u^{(3)}(y)R_x^{(6)}(y)|_0^1 \\ & + u^{(2)}(y)R_x^{(7)}(y)|_0^1 - u^{(1)}(y)R_x^{(8)}(y)|_0^1 \\ & + u(y)R_x^{(9)}(y)|_0^1 - \int_0^1 u(y)R_x^{(10)}(y)dy. \end{aligned} \tag{2.7}$$

Since, $u(y) \in W_2^5[0,1]$, so $u(0) = 0, u^{(1)}(0) = 0, u(1) = 0$.

If

$$\begin{cases} R_x^{(8)}(1) = 0, R_x^{(7)}(1) = 0, R_x^{(2)}(0) - R_x^{(7)}(0) = 0, \\ R_x^{(6)}(1) = 0, R_x^{(5)}(1) = 0, R_x^{(3)}(0) + R_x^{(6)}(0) = 0, \\ R_x^{(4)}(0) - R_x^{(5)}(0) = 0, \end{cases} \tag{2.8}$$

then Eq. (2.7) implies that

$$\langle u(y), R_x(y)\rangle = \int_0^1 (-1)u(y)R_x^{(10)}(y)dy.$$

For all $x \in [0,1]$, if $R_x(y)$ also satisfies

$$(-1)R_x^{(10)}(y) = \delta(y - x), \tag{2.9}$$

then

$$\langle u(y), R_x(y)\rangle = u(x). \tag{2.10}$$

When $y \neq x$ characteristic equation of Eq. (2.9) is given by $\lambda^{10} = 0$, then the characteristic values λ can be determined whose multiplicity is 10. The RK $R_x(y)$ can be defined as

$$R_x(y) = \begin{cases} \sum_{i=0}^{9} a_i y^i, & y \leqslant x, \\ \sum_{i=0}^{9} b_i y^i, & y > x. \end{cases} \tag{2.11}$$

Also let $R_x(y)$ satisfy

$$R_x^{(k)}(x + 0) = R_x^{(k)}(x - 0), \quad k = 0, 1, 2, \ldots, 8 \tag{2.12}$$

and integrate Eq. (2.9) from $x - \varepsilon$ to $x + \varepsilon$ with respect to y and $\varepsilon \to 0$. Using jump degree of $R_x^{(9)}(y)$ at $y = x$, gives

$$R_x^{(9)}(x - 0) - R_x^{(9)}(x + 0) = 1. \tag{2.13}$$

The coefficients $a_i s$ and $b_i s (i = 0, 1, 2, 3, \ldots, 9)$ can be determined from Eqs. (2.5), (2.8), (2.12) and (2.13) hence reproducing kernel obtained at $y \leqslant x$ is

$$\begin{cases} \begin{aligned} k(x,y) = \tfrac{1}{101500}[& y^2(-90720(-1+x)x^2(10780+x(700 \\ & +x(70+x(-56+x(28+(-8+x)x))))) \\ & -10080(-1+x)x^2(-90720+x(700 \\ & +x(70+x(-56+x(28+(-8+x)x)))))y \\ & -630(-1+x)x^2(-90720+x(-100800 \\ & +x(70+x(-56+x(28+(-8+x)x)))))y^2 \\ & -126(-1+x)x^2(-90720+x(-100800 \\ & +x(70+x(-56+x(28+(-8+x)x)))))y^3 \\ & +84(-1+x)x^2(-90720+x(700 \\ & +x(70+x(-56+x(28+(-8+x)x)))))y^4 \\ & -36(-1+x)x^2(10780+x(700 \\ & +x(70+x(-56+x(28+(-8+x)x)))))y^5 \\ & +9(-1+x)x(101500+x(10780 \\ & +x(700+x(70+x(-56+x(28 \\ & +(-8+x)x))))))y^6 - (-101500 \\ & +x^2(90720+x(10080+x(630+x(126 \\ & +x(-84+x(36+(-9+x)x)))))))y^7)] \end{aligned} \end{cases}$$

and for $x < y$, reproducing kernel $k(y,x)$ is given by

$k(y,x) = k(x,y)$. □

3. RK method for three-point BVP

3.1. Construction of RK satisfying three-point boundary conditions

RKM cannot be used directly to solve fourth order three-point boundary value problems (BVPs), since there is no method of obtaining reproducing kernel (RK) satisfying three-point boundary conditions, so the aim of this work is to fill this gap. A method for obtaining RK satisfying three-point boundary conditions is proposed so that RKM can be used to solve fourth order three-point BVPs.

3.1.1. Reproducing kernel space $W_\alpha^5[0,1]$

A reproducing kernel space $W_\alpha^5[0,1]$ is constructed in which every function satisfies, $u(0) = 0, u^{(1)}(0) = 0, u(\alpha) = 0, u(1) = 0$, where $\alpha \in]0,1[$. $W_\alpha^5[0,1]$ is defined as,

$W_\alpha^5[0,1] = \{u(x) | u(x) \in W_2^5[0,1], u(\alpha) = 0\}$.

RK of $W_\alpha^5[0,1]$ can be determined using the following Theorem

Theorem 3.1. *The reproducing kernel $K_\alpha(x,y)$ of $W_\alpha^5[0,1]$ is given as*

$$K_\alpha(x,y) = R_x(y) - \frac{R_x(\alpha)R_\alpha(y)}{R_\alpha(\alpha)} \tag{3.1}$$

Proof. Clearly, not all elements of $W_2^5[0,1]$ vanish at α. This implies that $R_\alpha(\alpha)$ is not zero. It is easy to see that $K_\alpha(x,\alpha) = K_\alpha(\alpha,y) = 0$, and therefore, $K_\alpha(x,y) \in W_\alpha^5[0,1]$. For all $u(y) \in W_\alpha^5[0,1], u(\alpha) = 0$. It follows that

$$\begin{aligned} \langle u(y), K_\alpha(y,x)\rangle &= \left\langle u(y), R_x(y) - \frac{R_x(\alpha)R_\alpha(y)}{R_\alpha(\alpha)} \right\rangle \\ &= \langle u(y), R_x(y)\rangle - \left\langle u(y), \frac{R_x(\alpha)R_\alpha(y)}{R_\alpha(\alpha)} \right\rangle \\ &= \langle u(y), R_x(y)\rangle - \frac{R_x(\alpha)u(\alpha)}{R_\alpha(\alpha)} \\ &= \langle u(y), R_x(y)\rangle = u(x). \end{aligned}$$

Hence $K_\alpha(x,y)$ satisfies reproducing property. Thus, $K_\alpha(x,y)$ is the reproducing kernel of $W_\alpha^5[0,1]$ and the proof is complete. □

3.2. Application of RKM

The linear fourth order three-point boundary value problem can be considered, as

$$\begin{cases} a_0(x)u^{(4)}(x) + a_1(x)u^{(3)}(x) + a_2(x)u^{(2)}(x) + a_3(x)u^{(1)}(x) \\ +a_4(x)u(x) = h(x), \ 0 \leqslant x \leqslant 1 \\ u(0) = \gamma_0, \quad u^{(1)}(0) = \gamma_1, \quad u(\alpha) = \gamma_2, \quad u(1) = \gamma_3. \end{cases} \tag{3.2}$$

Let bounded linear operator $L : W_\alpha^5[0,1] \to W_2^1[0,1]$ be defined as

$L = a_0(x)\frac{d^{(4)}}{dx} + a_1(x)\frac{d^3}{dx} + a_2(x)\frac{d^2}{dx} + a_3(x)\frac{d}{dx} + a_4(x)$, then Eq. (3.2) can be transformed into equivalent operator equation, as

$$\begin{cases} Lu(x) = h(x),\ 0 \leqslant x \leqslant 1, \\ u(0) = \gamma_0, \quad u^{(1)}(0) = \gamma_1, \quad u(\alpha) = \gamma_2, \quad u(1) = \gamma_3. \end{cases} \tag{3.3}$$

The inverse operator L^{-1} can be determined using RKM presented by Cui and Geng (2007), Geng and Cui (2007) and Cui and Lin (2009).

Let $\varphi_i(y) = \overline{K}(x_i, y), i \in N$, where $\overline{K}(x_i, y) \in W_2^1[0,1]$ is the reproducing kernel of $W_2^1[0,1]$, and $\psi_i(x) = L^*\varphi_i(x), i \in N$, where L^* is the adjoint operator of L. To orthonormalize the sequence $\{\psi_i(x)\}_{i=1}^{\infty}$, in the reproducing kernel space $W_\alpha^5[0,1]$, Gram Schmidt orthogonalization process can be used as follows

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad i = 1, 2, 3, \ldots \tag{3.4}$$

where β_{ik} is orthogonal coefficient.

Theorem 3.2.1. $\psi_i(x) = L_y K_\alpha(x, y)|_{y=x_i}$.

Proof. As

$$\psi_i(x) = L^*\varphi_i(x) = \langle L^*\varphi_i(y), K_\alpha(x, y)\rangle = \langle \varphi_i(y), L_y K_\alpha(x, y)\rangle$$
$$= \langle \bar{K}(x_i, y), L_y K_\alpha(x, y)\rangle = L_y K_\alpha(x, y)|_{y=x_i}. \qquad \square$$

Theorem 3.2.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_\alpha^5[0,1]$.*

Proof. Consider $< \langle u(x), \psi_i(x) >= 0$ which implies

$$\langle u(x), L^*\varphi_i(x)\rangle = 0 \Rightarrow \langle Lu(x), \varphi_i(x)\rangle = 0 \Rightarrow \langle Lu(x), \bar{K}(x_i, x)\rangle = 0 \Rightarrow Lu(x_i) = 0.$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$, so $Lu(x) \equiv 0$, which implies that $L^{-1}Lu(x) = 0$ and $u(x) \equiv 0$ from the existence of L^{-1}. $\square$

Theorem 3.2.3. *If $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$, and the solution of Eq. (3.2) is unique, $\forall u(x) \in W_\alpha^5[0,1]$, the series is convergent in the norm of $\|.\|_{W_\alpha^5}$. If $u(x)$ is the exact solution of Eq. (3.2), then it has the form:*

$$u(x) = L^{-1}h(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} h(x_k)\bar{\psi}_i(x).$$

Proof. Since $u(x) \in W_\alpha^5[0,1]$, so it can be expanded in the form of Fourier series about normal orthogonal system $\{\psi_i(x)\}_{i=1}^{\infty}$ as

$$u(x) = \sum_{i=1}^{\infty} \langle u(x), \bar{\psi}_i(x)\rangle \bar{\psi}_i(x). \tag{3.5}$$

Since the space $W_\alpha^5[0,1]$ is Hilbert space, so the series $\sum_{i=1}^{\infty}\langle u(x), \bar{\psi}_i(x)\rangle \bar{\psi}_i(x)$ is convergent in the norm of $\|.\|_{W_\alpha^5}$. It can be written as

$$u(x) = \sum_{i=1}^{\infty}\left\langle u(x), \sum_{k=1}^{i}\beta_{ik}\psi_k(x)\right\rangle \bar{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), \psi_k(x)\rangle \bar{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), L^*\varphi_k(x)\rangle \bar{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lu(x), \varphi_k(x)\rangle \bar{\psi}_i(x).$$

If $u(x)$ is the exact solution of Eq. (3.2) and $Lu(x) = h(x)$, then

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle h(x), \varphi_k(x)\rangle \bar{\psi}_i(x).$$

By applying reproducing property, it can be written as

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} h(x_k)\bar{\psi}_i(x),$$

which completes the proof. The approximate solution $u(x)$ is given by

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} h(x_k)\bar{\psi}_i(x). \qquad \square \tag{3.6}$$

Theorem 3.2.4. *For each $u(x) \in W_\alpha^5[0,1]$ and ε_n is the error between the approximate solution $u_n(x)$ and exact solution $u(x)$. Let $\varepsilon_n^2 = \|u(x) - u_n(x)\|^2$, then sequence $\{\varepsilon_n\}$ is monotone decreasing and $\varepsilon_n \to 0$ $(n \to \infty)$.*

Proof. Given

$$\varepsilon_n^2 = \|u(x) - u_n(x)\|^2 = \left\|\sum_{i=n+1}^{\infty}\langle u(x), \bar{\psi}_i(x)\rangle \bar{\psi}_i(x)\right\|^2$$
$$= \sum_{i=n+1}^{\infty}(\langle u(x), \bar{\psi}_i(x)\rangle)^2$$

$$\varepsilon_{n-1}^2 = \|u(x) - u_{n-1}(x)\|^2 = \left\|\sum_{i=n}^{\infty}\langle u(x), \bar{\psi}_i(x)\rangle \bar{\psi}_i(x)\right\|^2$$
$$= \sum_{i=n}^{\infty}(\langle u(x), \bar{\psi}_i(x)\rangle)^2.$$

Clearly $\varepsilon_{n-1} \geqslant \varepsilon_n$, consequently $\{\varepsilon_n\}$ is monotone decreasing in the sense of $\|.\|_{W_\alpha^5}$ and it is noted that the series is convergent in the norm of $\|.\|_{W_\alpha^5}$. Hence $\varepsilon_n \to 0$ $(n \to \infty)$. $\square$

4. Adomian decomposition method

The Adomian decomposition method was proposed by Adomian (1992, 1994) and Adomian and Rach (1994) for obtaining series solutions of algebraic, ordinary and partial differential equations, integral equations, integro-differential equations, etc. Such method has received a great deal of attention and has been applied to numerous problems. To solve non-linear fourth order three-point boundary value problem, the following decomposition method can be used as:

$$Lu(x) = f(x) + g(x, u(x)), \tag{4.1}$$

$$u(x) = L^{-1}f(x) + L^{-1}g(x, u(x)). \tag{4.2}$$

The ADM introduces the solution $u(x)$ and the nonlinear function $g(x, u)$ by infinite series, as

$$u(x) = \sum_{i=0}^{\infty} u_i(x) \tag{4.3}$$

and

$$g(x, u(x)) = \sum_{i=0}^{\infty} A_i(x), \tag{4.4}$$

where A_i is called Adomian polynomial and is defined by Adomian and Rach (1994) as

$$A_i = \frac{1}{i!}\left[\frac{d^i}{d\lambda^i} g\left(x, \sum_{i=0}^{\infty} \lambda^i u_i(x)\right)\right]_{\lambda=0}. \tag{4.5}$$

Substituting Eqs. (4.3) and (4.4) into Eq. (4.2), yields

$$\sum_{i=1}^{\infty} u_i(x) = L^{-1}f(x) + L^{-1}\sum_{i=1}^{\infty} A_i(x). \tag{4.6}$$

According to the ADM, the components $u_i(x)$ can be determined as

$$\begin{cases} u_0(x) = L^{-1}f(x), \\ u_{i+1}(x) = L^{-1}A_i(x), \quad i \geqslant 0. \end{cases} \tag{4.7}$$

By combining Adomian decomposition method and reproducing kernel method, Eq. (4.7) turns out to be

$$\begin{cases} u_0(x) = \sum_{j=1}^{\infty} B_{0j}\bar{\psi}_j(x), \\ \\ u_{i+1}(x) = \sum_{j=1}^{\infty} B_{(i+1)j}\bar{\psi}_j(x), i \geqslant 0, \end{cases} \tag{4.8}$$

where $B_{0j} = \sum_{k=1}^{j} \beta_{jk} f(x_k)$, $B_{ij} = \sum_{k=1}^{j} \beta_{jk} A_{i-1}(x_k), i \geqslant 1$. From Eq. (4.8), the components of $u_i(x)$ can be determined and hence the series solution $u(x)$ in Eq. (4.3) can be immediately obtained. For numerical purposes, the n-term approximation

$$U_n(x) = \sum_{i=0}^{n} u_i(x) \tag{4.9}$$

can be used to approximate the exact solution. Furthermore, the approximate solution $U_n^N(x)$ can be obtained by the N-term intercept of the exact solutions $u_i(x)$ and given by

$$U_n^N(x) = \sum_{i=0}^{n}\sum_{j=1}^{N} B_{ij}\bar{\psi}_j(x). \tag{4.10}$$

5. Numerical examples

In order to test the utility of the proposed method, four examples are considered in this section. All computations are performed using Mathematica 5.2.

Example 1. The singular linear fourth order three-point boundary value problem can be considered as

$$\begin{cases} x^4(1-x)u^{(4)}(x) + \frac{e^{\frac{x}{2}}}{2}u^{(3)}(x) + 2e^x \sin\sqrt{x}u^{(2)}(x) \\ \qquad +2u^{(1)}(x) + xu(x) = f(x), \qquad 0 \leqslant x \leqslant 1, \\ u(0) = 0, \quad u^{(1)}(0) = 1, \quad u(1) = \sinh(1), \quad u(\frac{3}{4}) = \sinh\left(\frac{3}{4}\right), \end{cases} \tag{5.1}$$

where $f(x) = x^4(1-x)\sinh(x) + \frac{e^{\frac{x}{2}}}{2}\cosh(x) + 2e^x \sin\sqrt{x}\sinh(x) + 2\cosh(x) + x\sinh(x)$. The exact solution of the problem (5.1) is $u(x) = \sinh(x)$. The numerical results are summarized in Table 1 and Figs. 1–3. From Figs. 1–3, it can easily be seen that the approximate solutions are in good agreement with exact solutions.

Example 2. Consider the following nonlinear fourth order three-point boundary value problem:

$$\begin{cases} u^{(4)}(x) - e^{-x}u^2(x) = 0, \ 0 \leqslant x \leqslant 1, \\ u(0) = 1, \quad u^{(1)}(0) = 1, \quad u(1) = e, \quad u(\frac{3}{4}) = e^{\frac{3}{4}}. \end{cases} \tag{5.2}$$

The exact solution of the problem (5.2) is $u(x) = e^x$. The combination of ADM and RKM is used to solve problem (5.2). The numerical results are summarized in Table 2 and Figs. 4–6. It can easily be seen from the Table 2 and Figs. 4–6 that the approximate solutions are in good agreement with exact solutions.

Example 3 (Mohyud-Din and Noor (2007)). Consider the following nonlinear fourth order three-point boundary value problem:

$$\begin{cases} u^{(4)}(x) + (u^{(2)}(x))^2 = sinx + sin^2(x), \qquad 0 \leqslant x \leqslant 1, \\ u(0) = 0, \quad u^{(1)}(0) = 1, \quad u(1) = sin1, \quad u(\frac{3}{4}) = sin\frac{3}{4}. \end{cases} \tag{5.3}$$

The exact solution of the problem (5.3) is $u(x) = sinx$. The combination of ADM and RKM is used to solve the problem (5.3). The numerical results are summarized in Table 3. The results of the problem are also compared with the method developed by Mohyud-Din and Noor (2007) in Table 3, which show that the present method is better.

Example 4. The nonlinear system of fourth order three-point boundary value problem can be considered, as :

Table 1 The numerical results when ($n = 6, 11, 51$).

x	Exact Solution	Relative error ($n = 6$)	Relative error ($n = 11$)	Relative error ($n = 51$)
0.0	0	0.0000	0.0000	0.0000
0.1	0.100167	4.73618E−06	4.35168E−07	4.27605E−10
0.2	0.201336	4.71631E−06	2.00644E−07	1.67976E−09
0.3	0.30452	2.3172E−06	4.40391E−08	2.77475E−09
0.4	0.410752	2.66477E−07	9.07671E−08	3.74118E−09
0.5	0.521095	9.27431E−07	1.88505E−07	4.65825E−09
0.6	0.636654	1.26386E−06	2.10614E−07	5.01064E−09
0.7	0.758584	5.7333E−07	1.0404E−07	2.77591E−09
0.8	0.888106	5.44023E−07	1.20919E−07	3.75365E−09
0.9	1.02652	1.05769E−06	2.7753E−07	1.01353E−08
1.0	1.1752	3.77883E−16	3.77883E−16	3.77883E−16

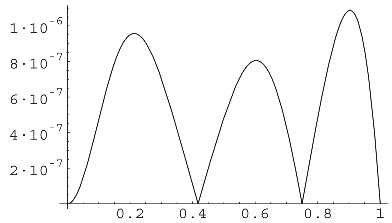

Figure 1 $|u - u_6|$.

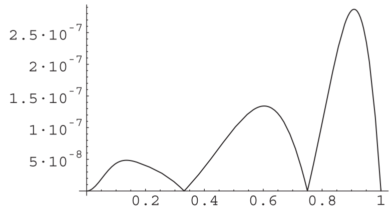

Figure 2 $|u - u_{11}|$.

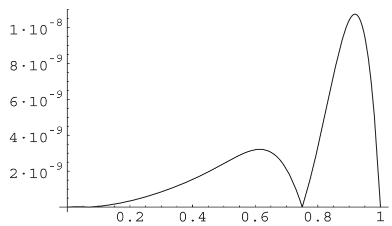

Figure 3 $|u - u_{51}|$.

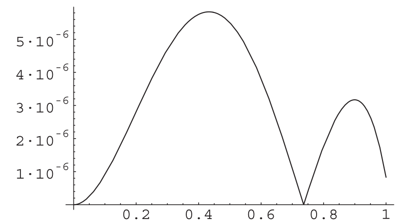

Figure 4 $|U - U_3^6|$.

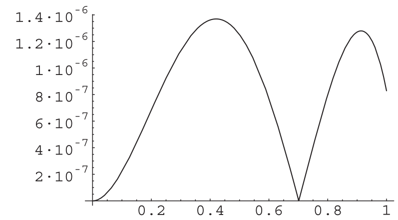

Figure 5 $|U - U_3^{11}|$.

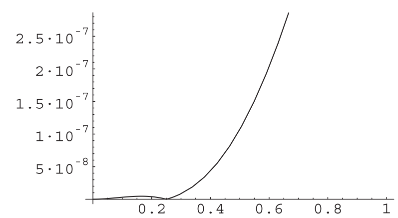

Figure 6 $|U - U_3^{51}|$.

Table 2 The numerical results when ($n = 3$).

x	Exact solution	Relative error ($N = 6$)	Relative error ($N = 11$)	Relative error ($N = 51$)
0.0	1	0.0000	0.0000	0.0000
0.1	1.10517	8.15367E−07	1.99301E−07	2.77828E−09
0.2	1.2214	2.30099E−06	6.18826E−07	3.23336E−09
0.3	1.34986	3.48073E−06	1.02349E−06	6.28379E−09
0.4	1.49182	3.85679E−06	1.23197E−06	2.89845E−08
0.5	1.64872	3.33694E−06	1.13206E−06	6.48551 E−08
0.6	1.82212	2.0949E−06	6.98202E−07	1.1154E−07
0.7	2.01375	5.42832E−07	1.09139E−08	1.6489E−07
0.8	2.22554	7.968E−07	7.22904E−07	2.19326E−07
0.9	2.4596	1.2893E−06	1.14967E−06	2.68284E−07
1.0	2.71828	4.5880E−16	3.3675E−16	6.6498E−16

$$u^{(4)}(x) = \begin{cases} u(x), & 0 \leqslant x \leqslant \frac{1}{4}, \ \frac{3}{4} \leqslant x \leqslant 1, \\ (u(x))^2 + f(x), & \frac{1}{4} \leqslant x \leqslant \frac{3}{4}, \\ u(0) = 0, \ u^{(1)}(0) = 0, \ u(1) = 1, \ u(\frac{1}{4}) = 2. \end{cases} \tag{5.4}$$

The exact solution of the problem (5.4) is

$$u(x) = \begin{cases} a_1x^2 + a_2x^3 + sinx, & 0 \leqslant x \leqslant \frac{1}{4}, \\ a_3x^2 + a_4x^3 + ln(1+x), & \frac{1}{4} \leqslant x \leqslant \frac{3}{4}, \\ -1 + a_5x^2 + a_6x^3 + e^x, & \frac{3}{4} \leqslant x \leqslant 1, \end{cases}$$

Table 3 The comparison between absolute error of the method developed and the method developed by Mohyud-Din and Noor (2007).

x	Absolute error in Mohyud-Din and Noor (2007)	Absolute error for ($N = 20, n = 3$)	Absolute error for ($N = 30, n = 3$)
0.0	9.592369E−14	0.0000	0.0000
0.1	7.7856E−8	4.248E−08	4.135E−09
0.2	2.723E−7	76.949E−08	5.783E−09
0.3	5.2489E−7	1.344E−08	3,956E−09
0.4	7.7730E−7	3.687E−08	2.843E−09
0.5	9.7145E−7	2.389E−08	7.933 E−09
0.6	1.0502E−6	7.133E−08	7.278E−09
0.7	9.6286E−7	8.677E−08	8.447E−09
0.8	6.8407E−7	2.355E−08	4.749E−09
0.9	2.7069E−7	2.678E−08	6.265E−09
1.0	1.5676E−13	0	0

where $a_1 = 37.6692$, $a_2 = -38.5107$, $a_3 = 37.1802$, $a_4 = -38.8985$, $a_5 = 38.1373$, $a_6 = -38.8305$ and $f(x) = -6(1+x)^4 - (38.1373x^2 - 38.8305x^3 + Ln(1+x))^2$. The combination of ADM and RKM is used to solve problem 4. The numerical results are summarized in Table 4. It is evident from Table 4 that the results are encouraging.

Table 4 The numerical results when ($n = 3$).

x	Absolute error ($N = 30$)	Absolute error ($N = 50$)
0.0	0	0.0000
0.1	4.46E−07	5.74E−08
0.2	2.74E−07	3.92E−08
0.3	8.24E−06	2.88E−07
0.4	9.95E−06	6.09E−07
0.5	3.22E−06	4.87E−07
0.6	2.86E−06	3.77E−07
0.7	1.37E−06	5.68E−07
0.8	9.57E−07	2.55E−08
0.9	5.69E−07	6.68E−08
1.0	0	0

6. Conclusion

Fourth order three point BVP (linear and nonlinear) and the system of fourth order three point BVP are determined using ADM and RKM. For the solution of linear fourth order three point boundary value problem reproducing kernel method is proposed and obtained encouraging results. The solution of non-linear fourth order three-point boundary value problem can be determined using standard Adomian decomposition method but this method has long calculation and complicated procedure to determine some unknown parameters. Due to this drawback a new computational method for the solution of non-linear fourth order three-point boundary value problem is proposed. This computational method is the combination of Adomian decomposition method and reproducing kernel method. Combination of these methods reduces the calculation and avoids the additional computation work in determining the unknown parameters, and this reduction has no effect in the accuracy of results. The comparison of the present method with the method Mohyud-Din and Noor (2007) available in literature also shows the efficiency of the method.

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