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ORIGINAL ARTICLE

New interaction solutions to the combined KdV–mKdV equation from CTE method

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Abstract The consistent tanh expansion (CTE) method is developed for the combined KdV–mKdV equation. The combined KdV–mKdV equation is proved to be CTE solvable. New exact interaction solutions such as soliton–cnoidal wave solutions, soliton–periodic wave solutions for the combined KdV–mKdV equation are given out analytically and graphically.

1. Introduction

As well known, how to find abundant more exact solutions for nonlinear systems, especially interaction solutions, is one of the most important aspects in the soliton theory. Many integrable properties, such as the multi-soliton solutions, Darboux transformation, symmetry reduction, Hirota bilinear form, homogeneous balance method, etc, are studied extensively after efforts of mathematicians and physicists. But for other integrable systems, it is still very difficult to find the interaction solutions among different types of excitations because there are no universal formulae to construct all the possible interaction wave solutions. However, according to the results of the symmetry reduction with nonlocal symmetries, Lou proposed the consistent tanh expansion method (CTE) recently (Lou, 2015), which is a more generalized but a much simpler method to find new interaction solutions between a soliton and other types of nonlinear excitations such as the soliton-resonant solution, soliton–cnoidal waves and soliton–perodic waves. Many new interaction solutions for nonlinear systems, for instance, the Boussinesq equation, dispersive water wave equation, Boussinesq-Burgers equation, break soliton equation, nonlinear Schrödinger equation and modified Kadomtsev-Petviashvili equation, are discussed in detail (Alam and Akbar, 2015; Bekir, 2009; Hu et al., 2012; Lou et al., 2014; Ren, 2015).

In this paper, we focus on the combined KdV–mKdV equation, which is also known as the Gardner equation

$$u_t + 2auu_x - 3u^2u_x + u_{xxx} = 0, \qquad (1)$$

where a is a constant. Eq. (1) is widely used in various fields of physics, such as solid-state physics, plasma physics, fluid physics and quantum field theory (Fu et al., 2004; Miura, 1997; Xu et al., 2003). The solitary solutions, traveling wave solutions, quasi-periodic solutions and symmetries for the combined KdV–mKdV Eq. (1) have been studied by means of the extended mapping method, extended tanh expansion method and new Riccati equation expansion method (Bekir, 2009; Huang and Zhang, 2006 Sirendaoerji, 2006; Zhao et al.,

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2006). The generalization form in (2+1)-dimensions of the combined KdV–mKdV equation has been discussed in Dai and Wang (2014), Geng and Cao (2001), Krishnan and Peng (2006), Zhang and Lin (1995), and Zhou and Ma (2000).

The paper is organized as follows. In Section 2, the CTE method will be defined briefly. New different interaction solutions including soliton–cnoidal waves, soliton–periodic waves solutions are obtained in Section 3. The detailed structures of new interaction solutions among nonlinear excitations are also given out graphically. The last section is devoted to summary and discussion.

2. Consistent tanh expansion for the combined KdV–mKdV equation

For a given nonlinear evolution system,

$$P(\mathbf{x},t,\mathbf{u})=0,\quad \mathbf{x}=\mathbf{x}(x_1,x_2,\ldots,x_n),\quad \mathbf{u}=\mathbf{u}(u_1,u_2,\ldots,u_n),\tag{2}$$

we aim to look for the possible truncated expansion solution

$$u=\sum_{j=0}^{n}u_j(x,t)\tanh^j(w),\tag{3}$$

where $w(x,t)$ is a function to be determined, n should be determined from the leading order analysis of Eq. (2) and all the expansion coefficients $u_j(x,t)$ will be determined by vanishing the coefficients of different powers of $\tanh(w)$ after substituting Eq. (3) into Eq. (2). If the system of u_j and w is consistent, we call the nonlinear system (2) is CTE solvable. In order to balance the nonlinear term u^2u_x and the dispersive term u_{xxx}, we have $3n+1=n+3$ and it is easy to find $n=1$, which is very similar to the leading order analysis of the Painlevé test for the nonlinear differential equation. We can seek the following truncated tanh expansion

$$u=u_0+u_1\tanh(w).\tag{4}$$

Substituting Eq. (4) into Eq. (1) yields

$$\begin{aligned}&3u_1w_x(u_1^2-2w_x^2)\tanh^4(w)+(6u_1w_xw_{xx}-2au_1^2w_x-3u_1^2u_{1x}\\&+6u_0w_xu_1^2+6u_{1x}w_x^2)\tanh^3(w)+[2au_1u_{1x}-3u_1^2(u_{0x}+u_1w_x)\\&-u_1w_t+8u_1w_x^3-3u_{1x}w_{xx}-2au_0u_1w_x+3u_0^2u_1w_x-u_1w_{xxx}\\&-6u_0u_1u_{1x}-3u_{1xx}w_x]\tanh^2(w)+[u_{1t}+u_{1xxx}+2au_0u_{1x}\\&-6u_1w_xw_{xx}+2au_1^2w_x+2au_1u_{0x}-6u_0u_1u_{0x}-6u_0u_1^2w_x\\&-3u_0^2u_{1x}-6u_{1x}w_x^2]\tanh(w)+u_{0t}+u_{0xxx}+3u_{1xx}w_x+u_1w_{xxx}\\&+2au_0u_{0x}+u_1w_t+2au_0u_1w_x-3u_0^2u_1w_x-2u_1w_x^3-3u_0^2u_{0x}\\&+3u_{1x}w_{xx}=0.\end{aligned}$$

Then setting the coefficients of different powers of $\tanh^j(w)$ to zero and solving the undetermined functions u_0, u_1, w, we have

$$u_0=\frac{2aw_x-3\sqrt{2}\sigma w_{xx}}{6w_x},\qquad u_1=\sqrt{2}\sigma w_x,\quad \sigma^2=1,\tag{5}$$

and

$$12w_x^2-6S-6C-2a^2=0,\tag{6}$$

$$(12w_x^2-6S-6C)_x+\frac{w_{xx}}{w_x}(12w_x^2-6S-6C-2a^2)=0,\tag{7}$$

$$\begin{aligned}&\left[\frac{w_{xx}^2}{w_x^2}+2w_x^2-\frac{w_{xxx}}{w_x}\right](12w_x^2-6S-6C-2a^2)\\&-(12w_x^2-6S-6C-2a^2)_{xx}\\&-\frac{w_{xx}}{w_x}(12w_x^2-6S-6C-2a^2)_x=0,\end{aligned}\tag{8}$$

where

$$S=\frac{w_{xxx}}{w_x}-\frac{3}{2}\frac{w_{xx}^2}{w_x^2},\qquad C=\frac{w_t}{w_x},$$

are Möbius invariants. It is easy to see that the Eqs. (7) and (8) are satisfied automatically because of (6) and we know that the w-consistent condition is just Eq. (6). From Eqs. (7) and (8), we can also have Eq. (6). Here we just consider the simplest Eq. (6) about w as the consistent condition to avoid the complicated calculation. In other words, if w is a solution of (6), then

$$u=\frac{2aw_x-3\sqrt{2}\sigma w_{xx}}{6w_x}+\sqrt{2}\sigma w_x\tanh(w),\tag{9}$$

is also a solution of the combined KdV–mKdV Eq. (1). So the expression (9) can be regarded as a nonauto-Bäcklund transformation of (1). Once the solution of (6) is known, then the solution of (1) will be obtained directly from (9). Many more interesting concrete interaction solutions will be studied in detail in the next section.

3. New interaction solutions for combined KdV–mKdV equation

It has been pointed out that if the solution of w-equation (6) is known, one can obtain the explicit solutions of Eq. (1) from (9). However, it is difficult to find the general solution of (6) because of its complexity. In order to obtain the interaction solutions between solitons and cnoidal waves of Eq. (1), we consider the function w in the form

$$w=k_1x+\omega_1t+W(X),\quad X=k_2x+\omega_2t.\tag{10}$$

Substituting (10) into Eq. (6), we can find that $W_1(X)$ satisfies

$$W_{1X}^2=4W_1^4+C_3W_1^3+\frac{C_2}{3}W_1^2+\frac{C_1}{3}W_1+\frac{C_0}{3},\tag{11}$$

with

$$\begin{aligned}&W(X)_X=W_1(X),\\&C_2=\frac{6\omega_2+9C_3k_1k_2^2+2a^2k_2-72k_2k_1^2}{k_2^3},\\&C_1=\frac{9C_3k_1^2k_2^2+3k_2\omega_1+9k_1\omega_2+4a^2k_1k_2-96k_1^3k_2}{k_2^4},\\&C_0=\frac{k_1(2a^2k_1k_2-36k_1^3k_2+3k_2\omega_1+3C_3k_1^2k_2^2+3k_1\omega_2)}{k_2^5},\end{aligned}$$

and C_3 is an arbitrary constant. It is known that the general solutions of Eq. (11) can be expressed in terms of Jacobi elliptic functions. The concrete examples of the soliton–cnoidal wave interaction solutions will be discussed in the following paper.

Firstly, we assume the function w in the form

$$w=k_0x+\omega_0t+c_1E_f(\mathrm{sn}(k_1x+\omega_1t,m),m),\tag{12}$$

where E_f is the first incomplete elliptic integral and sn is the usual Jacobi elliptic sine function. Substituting (12) into Eq. (6) and setting the coefficients of different powers of Jacobi elliptic functions into zero, we will have two constant solutions. The first case is

$$\omega_0 = \frac{k_0(48k_0^2k_1 - 2a^2k_1 - 3\omega_1)}{3k_1}, \quad c_1 = \frac{k_0}{k_1}, \tag{13}$$

with k_0, k_1, m, ω_1, a being arbitrary constants. The second case is

$$\omega_0 = \frac{k_0(6k_0^2 + 18k_1^2c_1^2 - a^2)}{3}, \quad \omega_1 = 2c_1^2k_1^3 + 6k_0^2k_1 - \frac{1}{3}a^2k_1, \tag{14}$$

where c_1, k_0, k_1, m, a are five arbitrary constants. Then substituting (12) with Eqs. (13) or (14) into Eq. (9) respectively, we will obtain different types of the soliton–cnoidal wave interaction solutions for the combined KdV–mKdV Eq. (1). We omit the complicated expression of the soliton–cnoidal wave interaction solutions and only the structure is shown in Fig.1 by selecting the arbitrary constants as

$$a = -1, \quad \omega_1 = -3, \quad k_0 = -0.5, k_1 = 2, \quad m = 0.98, \quad \sigma = 1. \tag{15}$$

When the arbitrary constants are fixed as

$$c_1 = 0.5, \quad a = 1, \quad k_0 = -1, \quad k_1 = 2, \quad m = 0.98, \quad \sigma = 1, \tag{16}$$

we can obtain new soliton–cnoidal wave interaction solution for combined KdV–mKdV Eq. (1) given by (9), (12), (14) and (16) and the detailed structure is given in Fig. 2.

Secondly, in order to find more interesting soliton–cnoidal wave interaction solution, we also can restrict the function w as

$$w = k_2x + \omega_2t + c_2E_\pi[\mathrm{sn}(k_3x + \omega_3t, m), n, m], \tag{17}$$

where the function E_π is the third incomplete elliptic integral and m is the module of the Jacobi elliptic sine function. After

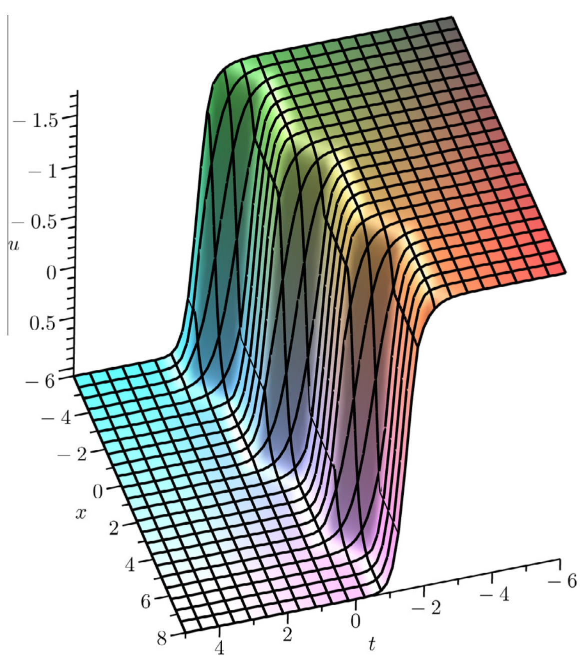

Fig. 1 First type of soliton–cnoidal wave interaction solution of u given by (9), (12), (13) and (15).

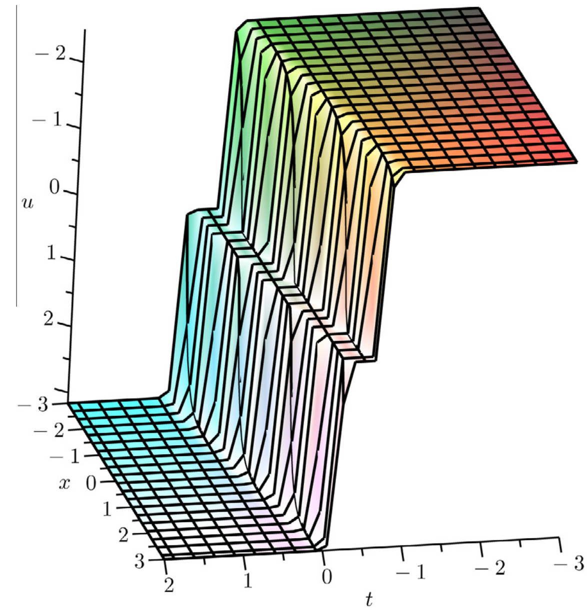

Fig. 2 Second type of soliton–cnoidal wave interaction solution of u given by (9), (12), (14) and (16).

substituting (17) into the consistent condition (6), we can arrive at the relations of constants

$$k_2 = -\frac{k_3(c_2^2 - n + 1)}{c_2}, \quad m = \frac{\sqrt{n(1-n)(c_2^2 - n + 1)}}{1-n}, \tag{18}$$

$$\omega_2 = \frac{1}{3c_2^3}\left([(18n-12)k_3^2 + a^2]c_2^4 - (n-1)[18k_3^2(n-1) + a^2]c_2^2 + 6(n-1)^3k_3^2\right) + \frac{2nk_3^2c_2^3}{1-n}, \tag{19}$$

$$\omega_3 = \frac{2nk_3^3c_2^2}{n-1} - \frac{k_3}{3c_2^2}\left[(12k_3^2(2n-1) + a^2)c_2^2 - 18k_3^2(n-1)^2\right], \tag{20}$$

with k_3, c_2, n, a being arbitrary constants. Substituting Eqs. (17)–(20) into Eq. (9) and selecting the arbitrary constants as

$$c_2 = 0.5, \quad n = 0.5, \quad k_3 = 2, \quad a = 1, \quad \sigma = 1, \tag{21}$$

we will have another interesting soliton–cnoidal wave interaction solution of the combined KdV–mKdV equation which is displayed in Fig. 3.

Lastly, it is known that the single soliton and periodic wave solutions expressed by hyperbolic functions and Jacobi elliptic functions have been studied in Zhao et al. (2006), Sirendaoerji (2006),Huang and Zhang (2006), Zhu (2014) by means of the extended tanh expansion method and Jacobi elliptic function expansion. It is clear that trivial solution of the consistent condition (6)

$$w = kx + \left(2k^3 - \frac{1}{3}a^3k\right)t,$$

leads to the single soliton for the combined KdV–mKdV equation. So we can obtain not only the usual soliton solution but also many new soliton–cnoidal wave interaction solutions in Figs. 1–3 for nonlinear integrable systems from the CTE method.

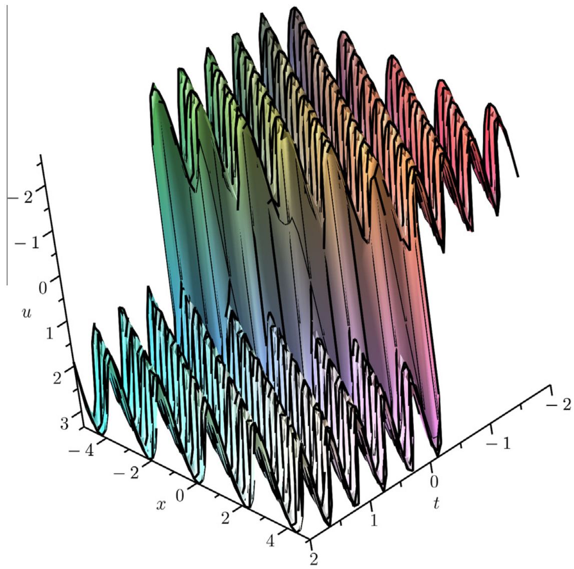

Fig. 3 Third type of soliton–cnoidal wave interaction solution of u given by (9) and (17)–(21).

4. Conclusion and discussion

In conclusion, we use a very simple CTE method to study the combined KdV–mKdV equation and it is proved to be CTE solvable. Many new exact interaction excitations such as the soliton–cnoidal wave interaction solutions, the soliton–periodic solutions are constructed directly from the CTE method by selecting different solutions to the consistent condition. These new interaction wave solutions are presented analytically and graphically with the proper constant selections. In general, for a CTE solvable system, we can obtain new different types of soliton–cnoidal wave solutions, which help us to learn the nonlinear system much better. The more interaction excitations from CTE method about other integrable systems, especially for the coupled integrable systems, such as the integrable coupled KdV system and the coupled integrable dispersionless system, will be worthy of study further.

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