Journal of the Association of Arab Universities for Basic and Applied Sciences (2016) **21**, 68–76

University of Bahrain

Journal of the Association of Arab Universities for Basic and Applied Sciences

www.elsevier.com/locate/jaaubas
www.sciencedirect.com

ORIGINAL ARTICLE

A simple harmonic balance method for solving strongly nonlinear oscillators

Md. Abdur Razzak

Dept. of Mathematics, Rajshahi University of Engineering and Technology (RUET), Kazla, Rajshahi-6204, Bangladesh

Received 27 June 2015; revised 27 August 2015; accepted 11 October 2015
Available online 28 November 2015

KEYWORDS

Nonlinear oscillation;
Harmonic balance method;
Truncation

Abstract In this paper, a simple harmonic balance method (HBM) is proposed to obtain higher-order approximate periodic solutions of strongly nonlinear oscillator systems having a rational and an irrational force. With the proposed procedure, the approximate frequencies and the corresponding periodic solutions can be easily determined. It gives high accuracy for both small and large amplitudes of oscillations and better result than those obtained by other existing results. The main advantage of the present method is that its simplicity and the second-order approximate solutions almost coincide with the corresponding numerical solutions (considered to be exact). The method is illustrated by examples. The present method is very effective and convenient method for solving strongly nonlinear oscillator systems arising in nonlinear science and engineering.

1. Introduction

Nonlinear oscillation problems are essential tool in physical science, mechanical structures, nonlinear circuits, chemical oscillation and other engineering research. Nonlinear vibrations of oscillation systems are modeled by nonlinear differential equations. It is very difficult to obtain periodic solutions of such nonlinear equations. There are several methods used to solve nonlinear differential equations. Among one of the widely used is perturbation method (Marion, 1970; Krylov and Bogoliubov, 1947; Bogoliubov and Mitropolskii, 1961; Nayfeh and Mook, 1979) whereby the nonlinear response is small. On the other hand, there are many methods (Amore and Aranda, 2005; Cheung et al., 1991; He, 2002) used to solve strongly nonlinear equations. The harmonic balance method (HBM) (Belendez et al., 2007; Mickens, 1996, 1984; Wu et al., 2006; Lim et al., 2005; Alam et al., 2007; Hosen et al., 2012) is another technique for solving strongly nonlinear equations. When a HBM is applied to the nonlinear equations for higher-order approximation, then a set of difficult nonlinear complex equations appear and it is very difficult to analytically solve these complex equations. In a recent article, Hosen et al. (2012) solved such nonlinear algebraic equations easily by using a truncation principle. Recently, many authors (Khan et al., 2011, 2012a,b, 2013a; Khan and Mirzabeigy, 2014; Saha and Patra, 2013; Yazdi et al., 2010; Yildirim et al., 2011a,b, 2012; Khan and Akbarzade, 2012; Akbarzade and Khan, 2012; Akbarzade and Khan, 2013) have studied strongly nonlinear oscillators. Khan et al. (2012a) used a coupling method combining homotopy and variational approach. Other authors (Nayfeh and Mook, 1979; Mickens, 2001; Hu and Tang, 2006; Lim and Wu, 2003) used HBM to solve some strongly nonlinear oscillators. But it is a very laborious procedure to obtain higher-order approximations using those methods (Nayfeh and Mook, 1979; Mickens, 2001; Hu and Tang,

E-mail address: ra_m@ruet.ac.bd
Peer review under responsibility of University of Bahrain.

http://dx.doi.org/10.1016/j.jaubas.2015.10.002
1815-3852 © 2015 University of Bahrain. Publishing services by Elsevier B.V.

2006; Lim and Wu, 2003). Fesanghary et al. (2009) obtained a new analytical approximation by variational iterative method (VIM); but the solution contains many harmonic terms. Many analytical techniques (Hosen et al., 2012; Mickens, 1996, 2001; Lim and Wu, 2003; Tiwari et al., 2005; Ozis and Yildirim, 2007; Ghadimi and Kaliji, 2013; Ganji et al., 2009; Zhao, 2009; Akbarzade and Farshidianfar, 2014; Khan et al., 2013a,b) have been used to solve strongly nonlinear oscillator systems having a rational force (such as Duffing-harmonic oscillator: $\ddot{x} + \frac{x^3}{1+x^2} = 0$ etc.) and irrational force (mass attached to a stretched wire: $\ddot{x} + x - \frac{x}{\sqrt{1+x^2}} = 0$ etc.). Hosen et al. (2012) solved the Duffing-harmonic oscillator by expanding the term $\frac{x^3}{1+x^2}$ into a polynomial form $\ddot{x} + x^3 - x^5 + \cdots = 0$. But this method (Hosen et al., 2012) is valid for small amplitude of oscillations and it is invalid to investigate the nonlinear oscillator $\ddot{x} + x - \frac{\lambda x}{\sqrt{1+x^2}} = 0$. On the contrary, other authors (Fesanghary et al., 2009; Khan et al., 2013a,b) have used different analytical techniques to solve these nonlinear oscillators $\ddot{x} + \frac{x^3}{1+x^2} = 0$; $\ddot{x} + x - \frac{\lambda x}{\sqrt{1+x^2}} = 0$ etc. without expanding. But their solution procedure for determining higher-order approximations of these nonlinear oscillators is not easy or straightforward and the results (obtained by second order approximation) are not more accurate compared with numerical results.

The purpose of this paper is to apply a simple factor on the strongly nonlinear oscillator systems having a rational and an irrational force and to obtain higher-order approximate frequencies and the corresponding periodic solutions by easily solving the sets of algebraic equations with complex nonlinearities. The trial solution (concern of this paper) is the same as that of Hosen et al. (2012). But the solution procedure is different from that of Hosen et al. (2012). To verify the accuracy of the present method, the two complicated nonlinear oscillators ($\ddot{x} + \frac{x^3}{1+x^2} = 0$; $\ddot{x} + x - \frac{\lambda x}{\sqrt{1+x^2}} = 0$) are chosen as examples. The method provides better result for both small and large amplitudes of oscillations. The significance of this present method is its simplicity, which not only provides a few harmonic terms, but also gives more accurate measurement than any other exiting solutions.

2. The methods

Let us consider the following general strongly nonlinear oscillator systems having a rational or an irrational force:

$$\ddot{x} + \omega_0^2 x + f(x) = 0, \tag{1}$$

with initial conditions

$$x(0) = A, \qquad \dot{x}(0) = 0, \tag{2}$$

where over dot denotes the derivatives with respect to t, A denotes the maximum amplitude, $f(x)$ is a nonlinear restoring-force function such that $f(-x) = -f(x)$ and $\omega_0 \geqslant 0$.

The approximate periodic solution of Eq. (1) is taken in the form similar to that of Hosen et al. (2012)

$$x_n(t) = A((1 - u_3 - u_5 \cdots)\cos\varphi + u_3 \cos 3\varphi + u_5 \cos 5\varphi + \cdots), \quad n = 0, 1, 2, \cdots, \tag{3}$$

where $\varphi = \omega t$, ω is an unknown angular frequency and $u_3, u_5, \cdots$ are constants which are to be further determined.

For the first-order approximation (putting $n = 0$ and $u_3 = u_5 = \cdots = 0$ in Eq. (3)), Eq. (3) becomes

$$x_0(t) = A\cos\varphi. \tag{4}$$

Eq. (4) also satisfied Eq. (2).

In this paper, Eq. (1) can be re-written as

$$\frac{\ddot{x} + \omega_0^2 x + f(x)}{1 + x_0^2} = 0. \tag{5}$$

Using Eqs. (3) and (4), we have the left-side the following Fourier series expansions:

$$\frac{\ddot{x} + \omega_0^2 x + f(x)}{(1 + x_0^2)} = c_1 \cos\varphi + c_3 \cos 3\varphi + c_5 \cos 5\varphi + \cdots, , \tag{6}$$

where

$$c_{2n-1} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{\ddot{x} + \omega_0^2 x + f(x)}{1 + x_0^2}\right)\cos(2n-1)\varphi\, d\varphi, \quad n = 1, 2, 3, \cdots, . \tag{7}$$

Substituting Eq. (6) for Eq. (5) and then equating the coefficients of the terms cos φ and cos 3φ, cos 5φ,..., we get a set of nonlinear algebraic equations whose solutions provide the unknown coefficients u_3, u_5, $\cdots$ together with the frequency, ω.

3. Examples

3.1. Example 1

Let us consider a one-dimensional, nonlinear Duffing-harmonic oscillator of the form (Mickens, 2001)

$$\ddot{x} + \frac{x^3}{1 + x^2} = 0, \tag{8}$$

with initial conditions

$$x(0) = A, \qquad \dot{x}(0) = 0. \tag{9}$$

Eq. (1) is an example of a conservative nonlinear oscillatory system having a rational form for the non-dimensional restoring force.

Mickens (2001) rearranged Eq. (8) as

$$(1 + x^2)\ddot{x} + x^3 = 0. \tag{10}$$

Applying the lowest order harmonic balance method to Eq. (10), Mickens (2001) obtained an approximate solution of this oscillator. For the higher-order approximation solutions, a set of complicated algebraic equations are involved and it is very difficult to analytically solve. On the other hand, Fesanghary et al. (2009) obtained higher-order solutions (containing up to ninth harmonic terms) from Eq. (10). To overcome these problems, approximation solutions (containing up to third harmonic terms) have been obtained by applying an easy approach to Eq. (10) which is based on HBM. In this article, nonlinear algebraic equations are solved by truncating higher order terms (followed partially by the principle rule presented in Hosen et al. (2012)).

Consider the second-order approximate periodic solution of Eq. (8) is of the form

$$x_1(t) = A((1 - u_3)\cos\varphi + u_3 \cos 3\varphi). \tag{11}$$

Therefore, the first-order approximation becomes

$$x_0(t) = A\cos\varphi. \tag{12}$$

In this paper, by dividing Eq. (10) by the factor $(1+x_0^2)$ and then substituting Eqs. (11) and (12) for Eq. (10) and also using Eqs. (6) and (7), finally, the approximate frequency ω and the approximate periodic solution can be found. It is noted that dividing by the factor $((1+x_0^2))$, a set of simple algebraic equations appears which contains lower order terms and these lower order terms make the solution rapidly converge. On the other hand, the results obtained in this paper give more accuracy than other existing results.

We re-write Eq. (8) in the form

$$\frac{(1+x^2)\ddot{x}+x^3}{1+x_0^2}=0. \tag{13}$$

Using Eqs. (11) and (12) in Eq. (13), we have the following Fourier series expansions:

$$\frac{(1+x^2)\ddot{x}+x^3}{1+x_0^2}=c_1\cos\varphi+c_3\cos 3\varphi+\cdots, \tag{14}$$

where

$$c_{2n-1}=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{(1+x^2)\ddot{x}+x^3}{1+x_0^2}\right)\cos(2n-1)\varphi\,d\varphi,\ n=1,2,3,\cdots. \tag{15}$$

The first two terms have been obtained as

$$c_1=b_0-\omega^2A+(b_1+\omega^2b_2)u_3+(b_3+\omega^2b_4)u_3^2+O(u_3^3),$$
$$c_3=d_0+(d_1+\omega^2d_2)u_3+O(u_3^2),$$

where

$$b_0=A-\frac{2}{A}+\frac{2}{A\sqrt{1+A^2}},\ b_1=\frac{24}{A^3}\left[1+\frac{A^2}{2}-\frac{A^4}{8}-\frac{1+A^2}{\sqrt{1+A^2}}\right],$$

$$b_2=\frac{16}{A^3}\left[-1-\frac{A^2}{2}+\frac{3A^4}{16}+\frac{1+A^2}{\sqrt{1+A^2}}\right],$$

$$b_3=\frac{96}{A^5}\left[-1-\frac{3A^2}{2}-\frac{3A^4}{8}+\frac{A^6}{16}+\frac{1+2A^2+A^4}{\sqrt{1+A^2}}\right],$$

$$b_4=\frac{16}{A^5}\left[38+49A^2+\frac{41A^4}{4}-\frac{11A^6}{8}-\frac{2(19+34A^2+15A^4)}{\sqrt{1+A^2}}\right],$$

$$d_0=\frac{8}{A^3}+\frac{2}{A}-\frac{8}{A^3(1+A^2)^{1/2}}-\frac{6}{A\sqrt{1+A^2}},$$

$$d_1=\frac{24}{A^5}\left[-4-5A^2-A^4+\frac{A^6}{8}+\frac{4+7A^2+3A^4}{\sqrt{1+A^2}}\right],$$

$$d_2=\frac{16}{A^5}\left[4+5A^2+A^4-\frac{11A^6}{16}-\frac{4+7A^2+3A^4}{\sqrt{1+A^2}}\right].$$

Here, we see that the above coefficients are not singular for A tending to zero. For example,

$$b_0=A-\frac{2}{A}+\frac{2}{A\sqrt{1+A^2}}=A-\frac{2A}{\sqrt{1+A^2}(\sqrt{1+A^2}+1)}.$$

So,

$$\lim_{A\to+0}b_0=0.$$

Substituting Eq. (14) for Eq. (13) and then equating the coefficients of the terms cos φ and cos 3φ equal to zeros, respectively, we obtain

$$b_0-\omega^2A+(b_1+\omega^2b_2)u_3+(b_3+\omega^2b_4)u_3^2+O(u_3^3)=0, \tag{16}$$

$$d_0+(d_1+\omega^2d_2)u_3+O(u_3^2)=0. \tag{17}$$

At first take $u_3=0$, Eq. (16) becomes

$$b_0-\omega^2A=0. \tag{18}$$

Solving Eq. (18), we obtain

$$\omega=\omega_1(A)=\sqrt{\frac{b_0}{A}}=\sqrt{1-\frac{2}{A^2}+\frac{2}{A^2\sqrt{1+A^2}}} \tag{19}$$

The first approximate frequency is given in Eq. (19) which was also obtained by Lim and Wu (2003).

It is obvious that the frequency as well as solution Eq. (11) gives better result when Eqs. (16) and (17) are truncated (Hosen et al. (2012)). Using the truncation rule of Hosen et al. (2012) in the Eqs. (16) and (17), we obtain the following results

$$b_0-\omega^2A+(b_1+\omega^2b_2)u_3+(b_3+\omega^2b_4)u_3^2/2=0, \tag{20}$$

$$d_0+(d_1+\omega^2d_2)u_3=0. \tag{21}$$

Eliminating ω from Eqs. (20) and (21), we obtain

$$-Ad_0+(b_2d_0-Ad_1-b_0d_2)u_3+(b_2d_1-b_1d_2+b_4d_0/2)u_3^2+\cdots=0. \tag{22}$$

Solving Eq. (22), we obtain

$$u_3=\frac{p(A)-\sqrt{q^2(A)-8Ad_0q(A)}}{2q(A)}, \tag{23}$$

$p(A)=2b_2d_0-2Ad_1-2b_0d_2,\ q(A)=2b_1d_2-b_4d_0-2b_2d_1.$

Solving Eq. (20), we obtain the second approximate frequency as

$$\omega(A)=\omega_2(A)=\sqrt{\frac{b_0-b_1u_3-b_3u_3^2/2}{A-b_2u_3+b_4u_3^2/2}}, \tag{24}$$

where u_3 is given by Eq. (23).

Therefore, the second-order approximate periodic solution of Eq. (8) is

$$x_1(t)=A((1-u_3)\cos\omega t+u_3\cos 3\omega t), \tag{25}$$

where u_3 and ω respectively, are given by Eqs. (23) and (24).

In a similar way, a third-order approximate solution,

$$x_2(t)=A((1-u_3-u_5)\cos\omega t+u_3\cos 3\omega t+u_5\cos 5\omega t) \tag{26}$$

is found for which the related equations are obtained as

$$b_0-\omega^2A+(b_1+\omega^2b_2)u_3+(b_3+\omega^2b_4)u_3^2 + (b_5+\omega^2b_6)u_3u_5/2+(b_7+\omega^2b_8)u_5=0, \tag{27}$$

$$d_0+(d_1+\omega^2d_2)u_3+(d_3+\omega^2d_4)u_3^2+(d_5+\omega^2d_6)u_5=0, \tag{28}$$

$$r_0+(r_1+\omega^2r_2)u_3+(r_3+\omega^2r_4)u_3^2+(r_5+\omega^2r_6)u_5=0. \tag{29}$$

where b_0, b_1, b_2, b_3, b_4; d_0, d_1, d_2 are already defined above and

$$b_5=\frac{192}{A^7}\left[4+7A^2+3A^4+\frac{A^6}{8}+\frac{A^8}{32}-\frac{(4+9A^2+6A^4+A^6)}{(1+A^2)^{1/2}}\right],$$

$$b_6 = \frac{64}{A^7}\left[-140 - 237A^2 - 105A^4 - \frac{59A^6}{8} - \frac{A^8}{16} + \frac{(140 + 307A^2 + 206A^4 + 39A^6)}{(1 + A^2)^{1/2}}\right],$$

$$b_7 = \frac{24}{A^5}\left[-4 - 3A^2 - \frac{A^4}{8} + \frac{4 + 5A^2 + A^4}{(1 + A^2)^{1/2}}\right], b_8 = -2b_7/3,$$

$$d_3 = \frac{96}{A^7}\left[4 + 9A^2 + 6A^4 + \frac{7A^6}{8} - \frac{3A^6}{32} - \frac{4 + 11A^2 + 10A^4 + 3A^6}{(1 + A^2)^{1/2}}\right],$$

$$d_4 = \frac{32}{A^7}\left[-76 - 155A^2 - 94A^4 - \frac{101A^6}{8} + \frac{41A^6}{32} + \frac{76 + 193A^2 + 162A^4 + 45A^6}{(1 + A^2)^{1/2}}\right],$$

$$d_5 = \frac{24}{A^7}\left[16 + 24A^2 + 9A^4 + \frac{A^6}{2} - \frac{16 + 32A^2 + 19A^4 + 3A^6}{(1 + A^2)^{1/2}}\right],$$

$$d_6 = -2d_5/3,$$

$$r_0 = \frac{2}{A^5}\left[-16 - 12A^2 - A^4 + \frac{16 + 20A^2 + 5A^4}{(1 + A^2)^{1/2}}\right],$$

$$r_1 = \frac{24}{A^7}\left[16 + 28A^2 + 13A^4 + A^6 - \frac{16 + 36A^2 + 25A^4 + 5A^6}{(1 + A^2)^{1/2}}\right],$$

$$r_2 = -2r_1/3, r_3 = \frac{32}{A^9}\left[-48 - 132A^2 - 123A^4 - 42A^6 - 3A^8 + \frac{3A^{10}}{32} + \frac{48 + 156A^2 + 183A^4 + 90A^6 + 15A^8}{(1 + A^2)^{1/2}}\right],$$

$$r_4 = \frac{128}{A^9}\left[304 + 772A^2 + 667A^4 + 214A^6 + 15A^8 - \frac{19A^{10}}{32} - \frac{304 + 1015A^2 + 1015A^4 + 470A^6 + 75A^8}{(1 + A^2)^{1/2}}\right],$$

$$r_5 = \frac{24}{A^9}\left[-64 - 128A^2 - 80A^4 - 17A^6 - A^8 + \frac{A^{10}}{8} + \frac{64 + 160A^2 + 136A^4 + 45A^6 + 5A^8}{(1 + A^2)^{1/2}}\right],$$

$$r_6 = -25A - 2r_5/3.$$

Solving Eq. (27) for ω, we obtain the third approximate frequency as

$$\omega(A) = \omega_3(A) = \sqrt{\frac{b_0 + b_1u_3 - b_3u_3^2 + b_5u_3u_5/2 + b_7u_5}{A - b_2u_3 - b_4u_3^2 - b_6u_3u_5/2 - b_8u_5}}, \quad (30)$$

where u_3 and u_5 are given in the following

$$u_3 = \mu + \frac{(Ad_1 + b_0d_2 - b_2d_0)(Ad_5 + b_0d_6 - b_8d_0)r_0}{Ad_0^2(Ar_5 + b_0r_6 - b_8r_0)}\mu^2 + (b_1d_2/Ad_0 - b_4/A + \cdots)\mu^3 + \cdots, \quad (31)$$

$$u_5 = \frac{1 + (r_1/r_0 + b_0r_2/(Ar_0) - b_2/A)\mu}{(b_8/A - r_5/r_0 - b_0r_6/(Ar_0))} + \cdots,$$

$$\mu = \frac{1}{(b_2/A - d_1/d_0 - b_0d_2/(Ad_0))}. \quad (32)$$

3.2. Example 2

In dimensionless form, we consider a mass attached to the center of a stretched elastic wire has the equation of motion (Sun et al. (2007))

$$\ddot{x} + x - \frac{\lambda x}{\sqrt{1 + x^2}} = 0, \quad (33)$$

where over dots denote differentiation with respect to time t and $0 < \lambda \leqslant 1$.

The initial conditions are

$$x(0) = A, \ \dot{x}(0) = 0. \quad (34)$$

Eq. (33) is an example of conservative nonlinear oscillatory system having an irrational elastic form for the restoring force.

Eq. (33) can be re-written as

$$\sqrt{1 + x^2}(\ddot{x} + x) - \lambda x = 0 \quad (35)$$

In this paper, we re-write Eq. (35) in the following form

$$\frac{\sqrt{1 + x^2}(\ddot{x} + x) - \lambda x}{\sqrt{1 + x_0^2}} = 0. \quad (36)$$

Using Eqs. (11) and (12) in Eq. (36), we have the following Fourier series expansions:

$$\frac{\sqrt{1 + x^2}(\ddot{x} + x) - \lambda x}{\sqrt{1 + x_0^2}} = c_1 \cos\varphi + c_3 \cos 3\varphi + \cdots, \quad (37)$$

where

$$c_{2n-1} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{(1 + x^2)\ddot{x} + x^3}{1 + x_0^2}\right)\cos(2n-1)\varphi\, d\varphi, n = 1, 2, 3, \cdots. \quad (38)$$

The first two terms have been obtained from Eq. (38) as

$$c_1 = e_0 - \omega^2 A + (e_2 + \omega^2 e_1)u_3 + O(u_3^2),$$

$$c_3 = p_0 + (p_2 + \omega^2 p_1)u_3 + O(u_3^2),$$

where

$$e_0 = A - \frac{4\lambda}{\pi A}\left[E(-A^2) - K(-A^2)\right],$$

$$e_1 = -\frac{4}{A^3}\left[2 + A^2 - A^3/2 - \frac{2 + A^2}{\sqrt{1 + A^2}}\right]$$

$$e_2 = -e_1 + \frac{16\lambda}{3\pi A^3}[(2 + A^2)E(-A^2) - 2(1 + A^2)K(-A^2)],$$

$$p_0 = \frac{4}{3\pi A^3}[(8 + A^2)K(-A^2) - (8 + A^2)E(-A^2)],$$

$$p_1 = \frac{8}{A^5}\left[4 + 5A^2 + A^3 - 5A^4/4 - \frac{4 + 7A^2 + 3A^4}{\sqrt{1 + A^2}}\right],$$

$$p_2 = -p_1 - 8A - \frac{16\lambda}{15\pi A^5}[(32 + 42A^2 + 7A^4)E(-A^2) - 2(16 + 29A^2 + 13A^4)K(-A^2)].$$

Substituting Eq. (37) for Eq. (36) and then equating the coefficients of the terms $\cos\varphi$ and $\cos 3\varphi$ equal to zeros, respectively, we obtain

$$e_0 - \omega^2 A + (e_2 + \omega^2 e_1)u_3 + O(u_3^2) = 0, \tag{39}$$

$$p_0 + (p_2 + \omega^2 p_1)u_3 + O(u_3^2) = 0. \tag{40}$$

At first take $u_3 = 0$, Eq. (39) becomes

$$e_0 - \omega^2 A = 0. \tag{41}$$

Solving Eq. (41), we obtain

$$\omega = \omega_1(A) = \sqrt{\frac{e_0}{A}} = \sqrt{1 - \frac{4\lambda}{\pi A^2}[E(-A^2) - K(-A^2)]} \tag{42}$$

The first approximate frequency is given in Eq. (42) which was also obtained by Sun et al. (2007).

Eliminating ω from Eqs. (39) and (40), we obtain

$$-Ap_0 + (e_1 p_0 - Ap_2 - e_0 p_1)u_3 + (p_2 e_1 - p_1 e_2)u_3^2 + \cdots = 0. \tag{43}$$

Solving Eq. (43), we obtain

$$u_3 = \frac{g(A) + \sqrt{g^2(A) + 4Ap_0 w(A)}}{2w(A)}, \tag{44}$$

where $g(A) = p_1 e_0 + Ap_2 - p_0 e_1$, $w(A) = e_1 p_2 - e_2 p_1$.

Solving Eq. (39), we obtain the second approximate frequency as

$$\omega(A) = \omega_2(A) = \sqrt{\frac{e_0 + e_2 u_3}{A - e_1 u_3}}, \tag{45}$$

where u_3 is given by Eq. (44).

Therefore, the second-order approximate periodic solution of Eq. (33) is

$$x_1(t) = A((1 - u_3)\cos\omega t + u_3 \cos 3\omega t), \tag{46}$$

where u_3 and ω respectively, are given by Eqs. (44) and (45).

4. Results and discussion

Based on the modified harmonic balance method (Hosen et al., 2012), an easy approach has been proposed to obtain higher-order approximate frequencies and the corresponding periodic solutions for both small and large values of amplitude of the strongly nonlinear oscillators having a rational and an irrational force. It has been already mentioned that the determination of second-order approximation is very difficult by methods Nayfeh and Mook (1979), Hu and Tang (2006), Lim and Wu (2003). In the present article, the higher-order approximate frequency as well as periodic solution especially second and third-order has been determined without any complicity and easily analytically solved (Eqs. (24), (30) and (45)).

We have calculated the second-order approximate frequencies for several amplitudes of oscillation (by Eq. (24)) of Eq. (8) and presented in Table 1. The results obtained in this paper are compared with those of others (calculated by Mickens, 2001; Lim and Wu, 2003; Tiwari et al., 2005; Ozis and Yildirim, 2007; Ghadimi and Kaliji, 2013; Khan et al., 2012a) all the

Table 1 Comparison of the approximate frequencies obtained by present method (Eq. (24)) with the exact frequency ω_e and other existing frequencies (those are obtained by Mickens, 2001; Lim and Wu, 2003; Tiwari et al. (2005), Ozis and Yildirim, 2007; Ghadimi and Kaliji, 2013; Khan et al., 2012a).

A	ω_e	Mickens	Lim and Wu	Tiw. et al.	Ozis and Yi	Ghadimi and Kaliji	Khan et al.	Present study	
		$Er(\%)$	$Er(\%)$	$Er(\%)$	$Er(\%)$	$Er(\%)$	$Er(\%)$	ω_2 $Er(\%)$	ω_3 $Er(\%)$
0.1	0.084389	0.086280	0.084256	0.086244	0.086268	0.084449	0.0757688	‘0.084389	0.084389
		2.240	0.158	2.198	2.227	0.071	10.215	0.000	0.000
0.2	0.166830	0.170664	0.166563	0.170393	0.170575	0.166964	0.150020	0.166826	0.166830
		2.298	0.160	2.136	2.245	0.080	10.076	0.002	0.000
0.4	0.319403	0.327327	0.318863	0.325513	0.326746	0.319757	0.288839	0.319359	0.319407
		2.481	0.169	1.913	2.299	0.111	9.569	0.014	0.001
0.6	0.449101	0.461084	0.448326	0.456392	0.459648	0.449777	0.409333	0.448990	0.449109
		2.668	0.173	1.624	2.345	0.151	8.855	0.025	0.002
0.8	0.554068	0.569495	0.553140	0.561440	0.567163	0.555136	0.509444	0.553895	0.554076
		2.784	0.168	1.331	2.364	0.193	8.054	0.031	0.002
1	0.636780	0.654654	0.635796	0.643594	0.651641	0.638285	0.590597	0.636572	0.636785
		2.807	0.155	1.070	2.333	0.236	7.252	0.033	0.000
2	0.847626	0.866026	0.847021	0.850651	0.862895	0.850963	0.811834	0.847505	0.847628
		2.171	0.071	0.357	1.801	0.394	4.222	0.042	0.000
3	0.919599	0.933257	0.919328	0.920897	0.931207	0.923295	0.895338	0.919574	0.919614
		1.485	0.0295	0.141	1.262	0.402	2.638	0.003	0.002
4	0.950856	0.960769	0.950730	0.951481	0.959428	0.954174	0.933926	0.950862	0.950875
		1.043	0.013	0.066	0.902	0.349	1.781	0.000	0.002
5	0.966976	0.974355	0.966913	0.967310	0.973431	0.969779	0.954642	0.966989	0.966992
		0.763	0.007	0.035	0.668	0.290	1.276	0.001	0.002
10	0.990916	0.993399	0.990912	0.990954	0.993144	0.992118	0.986940	0.990921	0.990921
		0.251	0.000	0.004	0.225	0.121	0.401	0.000	0.000

where $Er(\%)$ denotes the absolute percentage error.

Table 2a Comparison of the approximate frequencies obtained by present method (Eq. (45)) with the exact frequency ω_e and other existing frequencies (those frequencies obtained by Mickens, 1996a; Ganji et al., 2009; Zhao, 2009; Akbarzade and Farshidianfar, 2014; Khan et al., 2012a) when $\lambda = 0.5$.

A	ω_e	Mickens $Er(\%)$	Ganji et al. $Er(\%)$	Zhao $Er(\%)$	Akbarzade and Farshidianfar $Er(\%)$	Khan et al. $Er(\%)$	Present study $Er(\%)$
0.1	0.708423	0.707987	0.708424	0.86717 0	0.708431	0.708096	0.708423
		2.159	0.000	22.408	0.001	0.046	0.000
0.2	0.712259	0.710582	0.712271	0.870489	0.712390	0.710997	0.712259
		0.236	0.001	22.215	0.018	0.177	0.000
0.4	0.726126	0.720330	0.726271	0.882252	0.728011	0.721720	0.726125
		0.798	0.020	21.501	0.260	0.607	0.000
0.6	0.745140	0.734651	0.745683	0.897720	0.753326	0.737022	0.745145
		1.408	0.073	20.477	1.099	1.090	0.000
0.8	0.765907	0.751536	0.767072	0.913595	0.769064	0.754488	0.765903
		1.876	0.152	19.283	0.412	1.491	0.000
1	0.786171	0.769254	0.788075	0.927961	0.790009	0.772287	0.786165
		2.152	0.242	18.036	0.488	1.766	0.000
2	0.860447	0.843401	0.864865	0.969782	0.864890	0.843545	0.860451
		1.981	0.513	12.707	0.516	1.964	0.000
3	0.899904	0.887017	0.904671	0.984638	0.903592	0.885017	0.899922
		1.432	0.530	9.416	0.410	1.654	0.002
4	0.922727	0.912871	0.927153	0.990901	0.925721	0.910077	0.922749
		1.068	0.480	7.389	0.325	1.371	0.002
5	0.937317	0.929471	0.941285	0.994030	0.939793	0.926489	0.937338
		0.837	0.423	6.051	0.264	1.155	0.002
10	0.968102	0.964358	0.970480	0.998456	0.969373	0.962054	0.968113
		0.387	0.246	3.136	0.131	0.625	0.001

Table 2b Comparison of the approximate frequencies obtained by present method (Eq. (45)) with the exact frequency ω_e and other existing frequencies when $\lambda = 0.75$.

A	ω_e	Mickens $Er(\%)$	Ganji et al. $Er(\%)$	Zhao $Er(\%)$	Akbarzade and Farshidianfar $Er(\%)$	Khan et al. $Er(\%)$	Present study $Er(\%)$
0.1	0.502786	0.501865	0.502788	0.664804	0.502805	0.502095	0.502786
		0.183	0.000	32.224	0.003	0.138	0.000
0.2	0.510841	0.507336	0.510876	0.674494	0.511126	0.508213	0.510840
		0.686	0.007	32.036	0.056	0.515	0.000
0.4	0.539214	0.527553	0.539633	0.708046	0.543139	0.530449	0.539211
		2.163	0.0778	31.311	0.728	1.626	0.000
0.6	0.576587	0.556389	0.577983	0.750517	0.592663	0.561244	0.576575
		3.503	0.242	30.165	2.788	2.661	0.002
0.8	0.615781	0.589244	0.618545	0.792449	0.655744	0.595194	0.615756
		4.310	0.449	28.6901	6.490	3.343	0.004
1	0.652771	0.622597	0.656958	0.829156	0.660432	0.628649	0.652735
		4.622	0.642	27.021	1.174	3.695	0.005
2	0.780662	0.752986	0.788662	0.930630	0.788702	0.753788	0.780668
		3.545	1.025	19.211	1.030	3.443	0.000
3	0.844964	0.824742	0.853020	0.965092	0.851304	0.821921	0.845013
		2.393	0.954	14.217	0.750	2.727	0.006
4	0.881255	0.866025	0.888492	0.979408	0.886250	0.861890	0.881313
		1.728	0.821	11.138	0.567	2.197	0.007
5	0.904141	0.892119	0.910509	0.986516	0.908195	0.887659	0.904197
		1.330	0.704	9.111	0.448	1.823	0.006
10	0.951696	0.946033	0.955378	0.996522	0.953690	0.942570	0.951722
		0.595	0.387	4.710	0.210	0.959	0.003

results together with exact frequency have been shown in Table 1. Table 1 indicates that the approximate frequencies (concern of this paper) are better than those obtained by Mickens, 2001; Lim and Wu, 2003; Tiwari et al., 2005; Ozis and Yildirim, 2007; Ghadimi and Kaliji, 2013; Khan et al., 2012a.

Table 2c Comparison of the approximate frequencies obtained by present method (Eq. (45)) with the exact frequency ω_e and other existing frequencies when $\lambda = 0.95$.

A	ω_e	Mickens $Er(\%)$	Ganji et al. $Er(\%)$	Zhao $Er(\%)$	Akbarzade and Farshidianfar $Er(\%)$	Khan et al. $Er(\%)$	Present study $Er(\%)$
0.1	0.231367	0.228836	0.231391	0.323516	0.231436	0.229483	0.231367
		1.094	0.011	39.828	0.023	0.814	0.000
0.2	0.252549	0.243639	0.252836	0.354238	0.253476	0.246025	0.252547
		3.528	0.114	40.265	0.367	2.583	0.000
0.4	0.317642	0.293022	0.319674	0.447114	0.327109	0.300135	0.317594
		7.751	0.640	40.760	2.981	5.511	0.015
0.6	0.391035	0.354195	0.395577	0.547084	0.422197	0.364869	0.390906
		9.421	1.162	39.907	7.967	6.692	0.033
0.8	0.459947	0.416090	0.466860	0.634908	0.527257	0.428086	0.459761
		9.535	1.503	38.039	14.634	6.927	0.0404
1	0.520335	0.473633	0.529168	0.706124	0.534618	0.485141	0.520138
		8.975	1.697	35.706	2.745	6.764	0.038
2	0.709629	0.671950	0.721931	0.886069	0.721987	0.674087	0.709623
		5.310	1.734	24.864	1.742	5.009	0.000
3	0.797913	0.771310	0.809330	0.943366	0.807038	0.768077	0.798006
		3.334	1.431	18.229	1.144	3.739	0.012
4	0.846399	0.826640	0.856307	0.966748	0.853359	0.821520	0.846507
		2.335	1.171	14.219	0.822	2.940	0.0123
5	0.876561	0.861071	0.885117	0.978276	0.882101	0.855466	0.876659
		1.767	0.976	11.604	0.632	2.407	0.011
10	0.938333	0.931114	0.943122	0.994413	0.940956	0.926722	0.938376
		0.7693	0.510	5.977	0.280	1.238	0.005

Next, we have calculated the second-order approximate frequency (by Eq. (45)) of Eq. (33) with several amplitudes of oscillation and some different values of λ that are compared with numerical result and other existing results (those results obtained by Mickens, 1996; Ganji et al., 2009; Zhao, 2009; Akbarzade and Farshidianfar, 2014; Khan et al., 2012a) which are presented in Tables 2a–c. The absolute percentage errors of each method has been calculated and presented in all Tables 1 and 2c. The results of these Tables show that the approximate frequencies (concern by this paper) provide better results than those obtained by Mickens (1996), Ganji et al. (2009), Zhao (2009), Akbarzade and Farshidianfar (2014), Khan et al. (2012a). Thus, the present method provides better result than other existing results.

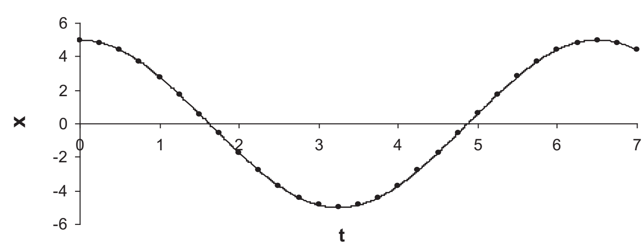

Figure 1b The present method solution of Eq. (8) has been presented (denoting b circles) when large value of amplitude $A = 5.0$, with initial conditions $[x(0) = 5.0, \dot{x}(0) = 0]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with present method.

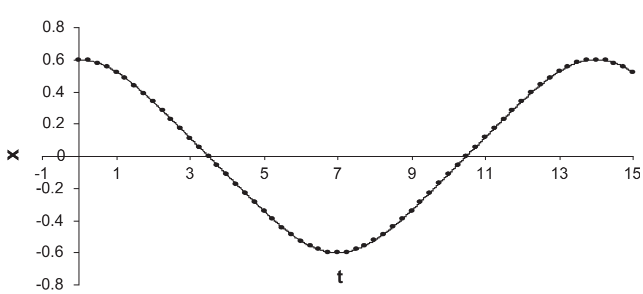

Figure 1a The present method solution of Eq. (8) has been presented (denoting by circles) when small value of amplitude $A = 0.6$, with initial conditions $[x(0) = 0.6, \dot{x}(0) = 0]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with present method.

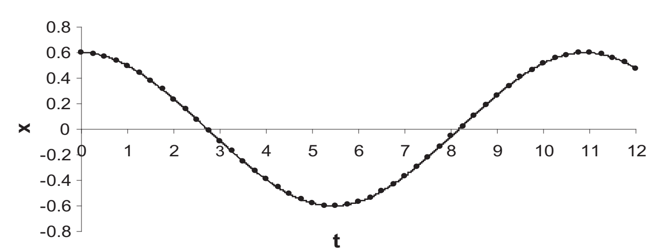

Figure 2a The present method solution of Eq. (33) has been presented (denoting by circles) when small value of amplitude $A = 0.6$ and $\lambda = 0.75$, with initial conditions $[x(0) = 0.6, \dot{x}(0) = 0]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with present method.

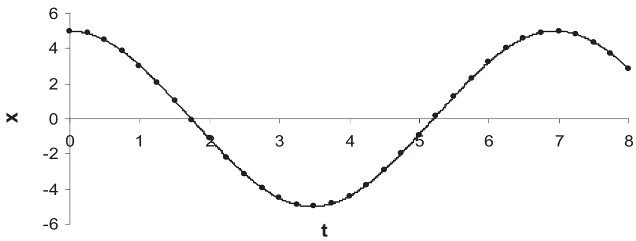

Figure 2b The present method solution of Eq. (33) has been presented (denoting by circles) when small value of amplitude $A = 5.0$ and $\lambda = 0.75$, with initial conditions $[x(0) = 5.0, \dot{x}(0) = 0]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with present method.

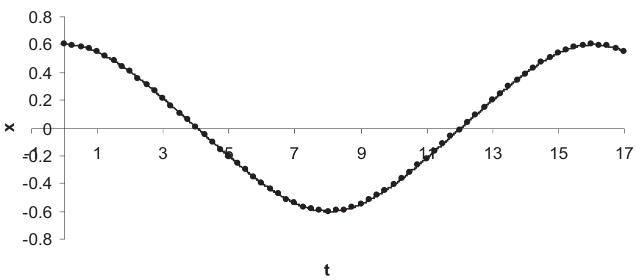

Figure 2c The present method solution of Eq. (33) has been presented (denoting by circles) when small value of amplitude $A = 0.6$ and $\lambda = 0.95$, with initial conditions $[x(0) = 0.6, \dot{x}(0) = 0]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with present method.

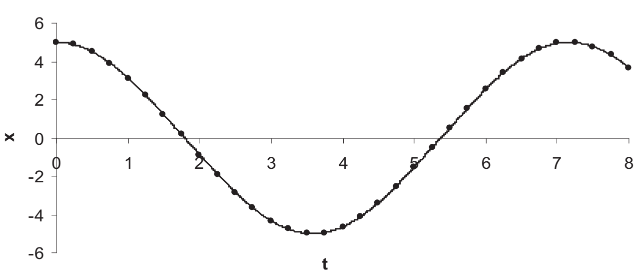

Figure 2d The present method solution of Eq. (33) has been presented (denoting by circles) when large value of amplitude $A = 5.0$ and $\lambda = 0.95$, with initial conditions $[x(0) = 5.0, \dot{x}(0) = 0]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with present method.

Next, the second-order approximate periodic solution of Eq. (8) has been determined by present method when $A = 0.6$ and $A = 5.0$ and respectively presented in Fig. 1a and b and each figure includes the corresponding numerical solution. In a similar way, the second-order approximate periodic solution of Eq. (33) has been determined by present method for small value and large values of amplitude which are presented in Fig. 2a–d and each figure includes the corresponding numerical solution.

Seeing all the figures, we observe that the present method solutions are nicely agreement with the corresponding numerical solutions for both small and large amplitudes of oscillations.

5. Conclusions

In present work, a simple harmonic balance approach has been presented to obtain higher-order approximations of strongly nonlinear oscillator systems having a rational and an irrational force. Recently, many analytical techniques have been developed to solve strongly nonlinear oscillators. But these techniques are very difficult to determine higher-order approximations because a set of complex nonlinear algebraic equations involve higher order terms and it is very difficult to solve these equations analytically. In this article, this limitation has been eliminated by applying a simple factor to the nonlinear oscillators. The solution procedure of the present approach is very simple involving a lower order term. Results obtained in this paper are compared with other existing results. As indicated, the error of the present method is much low than others. On the other hand, it makes the approximate solution rapidly converge. Thus, the present approach is an extremely effective and powerful method for solving strongly nonlinear oscillator systems arising in nonlinear science and engineering especially in vibration engineering.

Acknowledgments

The author is grateful to the reviewers for their helpful comments/suggestions in improving the manuscript.

References

Akbarzade, M., Farshidianfar, A., 2014. Application of the amplitude-frequency formulation to a nonlinear vibration system typified by a mass attached to a stretched wire. Int. Appl. Mech. 50 (4), 476–483.

Akbarzade, M., Khan, Y., 2012. Dynamic model of large amplitude non-linear oscillations arising in the structural engineering: analytical solutions. Math. Comput. Model. 55, 480–489.

Akbarzade, M., Khan, Y., 2013. Nonlinear dynamic analysis of conservative coupled systems of mass-spring via the analytical approaches. Arab. J. Sci. Eng. 38 (1), 155–162.

Alam, M.S., Haque, M.E., Hossian, M.B., 2007. A new analytical technique to find periodic solutions of nonlinear systems. Int. J. Nonlinear Mech. 42, 1035–1045.

Amore, P., Aranda, A., 2005. Improved Lindstedt–Poincare method for the solution of nonlinear problems. J. Sound Vib. 283 (3–5), 1115–1136.

Belendez, A., Pascual, C., Gallego, S., Ortuno, M., Neipp, C., 2007. Application of a modified He's homotopy perturbation method to obtain higher-order approximation of an $x^{\{1/3\}}$ force nonlinear oscillator. Phys. Lett. A 371, 421–426.

Bogoliubov, N.N., Mitropolskii, Yu.A., 1961. Asymptotic Methods in the Theory of Nonlinear Oscillations. Gordan and Breach, New York.

Cheung, Y.K., Mook, S.H., Lau, S.L., 1991. A modified Lindstedt–Poincare method for certain strongly nonlinear oscillators. Int. J. Nonlinear Mech. 26, 367–378.

Fesanghary, M., Pirbodaghi, T., Asghari, M., Sojoudi, H., 2009. A new analytical approximation to the duffing-harmonic oscillator. Chaos Solitons Fractals 42, 571–576.

Ganji, D.D., Malidarreh, N.R., Akbarzade, M., 2009. Comparison of energy balance period with exact period for arising nonlinear oscillator equations. Acta Appl. Math. 108, 353–362.
Ghadimi, M., Kaliji, H.D., 2013. Application of the harmonic balance method on nonlinear equations. World Appl. Sci. J. 22 (4), 532–537.
He, J.H., 2002. Modified Lindstedt-Poincare methods for some strongly nonlinear oscillations-I: expansion of a constant. Int. J. Nonlinear Mech. 37, 309–314.
Hosen, M.A., Rahman, M.S., Alam, M.S., Amin, M.R., 2012. An analytical technique for solving a class of strongly nonlinear conservative systems. Appl. Math. Comput. 218, 5474–5486.
Hu, H., Tang, J.H., 2006. Solution of a duffing-harmonic oscillator by the method of harmonic balance. J. Sound Vib. 294 (3), 637–639.
Khan, Y., Akbarzade, M., 2012. Dynamic analysis of non-linear oscillator equation arising in double-sided driven clamped microbeam-based electromechanical resonator. Z. Naturforsch. 67a, 435–440.
Khan, Y., Mirzabeigy, A., 2014. Improved accuracy of He's energy balance method for analysis of conservative nonlinear oscillator. Neural Comput. Appl. 25, 889–895.
Khan, Y., Madani, M., Yildirim, A., Abdou, M.A., Faraz, N., 2011. A new approach to van der pol's oscillator problem. Z. Naturforsch. 66a, 620–624.
Khan, Y., Akbarzade, M., Kargar, A., 2012a. Coupling of homotopy and variational approach for a conservative oscillator with strong odd-nonlinearity. Sci. Iran. 19 (3), 417–422.
Khan, Y., Leal, H.V., Faraz, N., 2012b. An efficient new iterative method for oscillator differential equation. Sci. Iran. 19 (6), 1473–1477.
Khan, Y., Yazdi, M.K., Askari, H., Saadatnia, Z., 2013a. Dynamic analysis of generalized conservative nonlinear oscillators via frequency amplitude formulation. Arab. J. Sci. Eng. 38 (1), 175–179.
Khan, Y., Fardi, M., Boroujeni, F.H., 2013b. Nonlinear oscillators with rational terms: a new semi-analytical technique. Sci. Iran. 20 (4), 1153–1160.
Krylov, N.N., Bogoliubov, N.N., 1947. Introduction to Nonlinear Mechanics. Princeton University Press, New Jersey.
Lim, C.W., Wu, B.S., 2003. A new analytical approach to the duffing-harmonic oscillator. Phys. Lett. A 311 (5), 365–377.
Lim, C.W., Lai, S.K., Wu, B.S., 2005. Accurate higher-order analytical approximate solutions to large-amplitude oscillating systems with general non-rational restoring force. J. Nonlinear Dyn. 42, 267–281.
Marion, J.B., 1970. Classical Dynamics of Particles and System. Harcourt Brace Jovanovich, San Diego, CA.
Mickens, R.E., 1984. Comments on the method of harmonic balance. J. Sound Vib. 94, 456–460.
Mickens, R.E., 1996. Oscillation in Planar Dynamic Systems. World Scientific, Singapore.
Mickens, R.E., 2001. Mathematical and numerical study of duffing-harmonic oscillator. J. Sound Vib. 244 (3), 563–567.
Nayfeh, A.H., Mook, D.T., 1979. Nonlinear Oscillations. John Wiley & Sons, New York.
Ozis, T., Yildirim, A., 2007. Determination of the frequency-amplitude relation for a duffing-harmonic oscillator by the energy balance method. Comput. Math. Appl. 54, 1184–1187.
Saha, R.S., Patra, A., 2013. Haar wavelet operational methods for the numerical solutions of fractional order nonlinear oscillatory van der pol system. Appl. Math. Comput. 220, 659–667.
Sun, W.P., Wu, B.S., Lim, C.W., 2007. Approximate analytical solutions for oscillation of a mass attached to a stretched elastic wire. J. Sound Vib. 300, 1042–1047.
Tiwari, S.B., Rao, B.N., Swamy, N.S., Sai, K.S., Nataraja, H.R., 2005. Analytical study on a duffing-harmonic oscillator. J. Sound Vib. 285, 1217–1222.
Wu, B.S., Sun, W.P., Lim, C.W., 2006. An analytical approximate technique for a class of strongly nonlinear oscillators. Int. J. Nonlinear Mech. 41, 766–774.
Yazdi., M. K., Khan, Y., Madani, M., Askari, H.,Saadatnia, Z., Yildirim, A., 2010. Analytical solutions for autonomous conservative nonlinear oscillator. Int. J. Non-linear Sci. Numer. Simu. 11 (11), 979-984.Yazdi, M.K., Khan, Y., Madani, M., Askari, H., Saadatnia, Z., Yildirim, Z., 2010. Analytical solutions for autonomous conservative nonlinear oscillator. Int. J. Non-linear Sci. Numer. Simu. 11 (11), 979–984.
Yildirim, A., Saadatnia, Z., Askari, H., Khan, Y., Yazdi, M.K., 2011a. Higher order approximate periodic solutions for nonlinear oscillators with the Hamiltonian approach. Appl. Math. Lett. 24 (12), 2042–2051.
Yildirim, A., Askari, H., Saadatnia, Z., Yazdi, M.K., Khan, Y., 2011b. Analysis of nonlinear oscillations of a punctual charge in the electric field of a charged ring via a Hamiltonian approach and the energy balance method. Comput. Math. Appl. 62, 486–490.
Yildirim, A., Askari, H., Yazdi, M.K., Khan, Y., 2012. A relationship between three analytical approaches for the nonlinear problems. Appl. Math. Lett. 25, 1729–1733.
Zhao, L., 2009. He's frequency-amplitude formulation for nonlinear oscillators with an irrational force. Comput. Math. Appl. 58, 2477–2479.