Generalized Order Statistics from $q$–Exponential Type- I Distribution and its Characterization

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Received Nov. 21, 2014, Revised Jan. 9, 2015, Accepted Jan. 16, 2015, Published 1 May 2015

Abstract: This article is concerned with $q$–exponential type-1 distribution. Recurrence relations for single and product moments of generalized order statistics have been derived from $q$–exponential type-1 distribution. Single and product moments of ordinary order statistics and upper $k$ records cases have been discussed as a special case from generalized order statistics.

Keywords: Generalized order statistics, Order statistics, Record values, Single and product moments, Recurrence relations, $q$–exponential type-1 distribution and characterization.

1. INTRODUCTION

Kamps (1995) introduced the concept of generalized order statistics (gos) as follows:

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variable (rv) with the df $F(x)$ and the pdf $f(x)$. Let $n \in \mathbb{N}$, $n \geq 2$, $k > 0$, $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n - 1\}$. Then $X(r, n, \mathbf{m}, k)$, $r = 1, 2, \ldots, n$ are called (gos) if their joint pdf is given by

$$k \left( \prod_{j=1}^{n} \gamma_j \right) \left( \prod_{i=1}^{n} [1 - F(x_i)]^m f(x_i) \right) [1 - F(x_n)]^{k-1} f(x_n) \quad (1.1)$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$ of $\mathbb{R}^n$.

The joint density of the first $r$ gos is given by

$$f_{X(1,n,\mathbf{m},k),\ldots,X(r,n,\mathbf{m},k)}(x_1, x_2, \ldots, x_r) = C_{r-1} \left( \prod_{i=1}^{r} [\bar{F}(x_i)]^m f(x_i) \right) \bar{F}(x_r)^{k+n-r+M_r-1} f(x_r) \quad (1.2)$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$.

Then it is called generalized order statistics of a sample from distribution with $df F(x)$.

The pdf of $r^{th}$ $m$–gos is given by [Kamps, 1995]:

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{r-1} f(x) g_m^{-1} [F(x)] \quad (1.3)$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, the $r^{th}$ and $s^{th}$ $m$–gos, $1 \leq r < s \leq n$, is
\[
f_{X(r,n,m,k),X(r,n,m,k)}(x,y) = \frac{C_{r-1}}{(r-1)!((x-r-1)!)^m} g_m^{r-1} \left[ F(x) \right]^{r-1} g_m^{r-1} \left[ f(y) \right]^{r-1} f(x) f(y), \quad \alpha \leq x < y \leq \beta \tag{1.4}
\]

where
\[
C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1),
\]
\[
h_m(x) = \begin{cases} \frac{1}{m+1} (1-x)^{m+1}, & m \neq -1 \\ -\log(1-x), & m = -1 \end{cases}
\]
and
\[
g_m(x) = \int_{0}^{x} (1-t)^m \, dt = h_m(x) - h_m(0), \quad x \in [0,1).
\]

We define the \( q \)-exponential distribution is a generalization of the exponential distribution. The main reason for introducing \( q \)-exponential model is the switching property of the exponential form to corresponding binomial expansion. We refer the reader to Seetha and Thomas (2012) for a comprehensive study on the properties of \( q \)-exponential distribution
\[
\lim_{q \to 0} \left[ 1 + (q-1)z \right]^{\frac{1}{q-1}} = e^{-z}, \quad 1 < q < 2
\]

The main properties of the \( q \)-exponential distribution as follows,

1. Exponential distribution is a special case
2. It has equi-dispersed data via shape parameter,
3. It allows for non-constant hazard rates

A random variable \( X \) is said to have \( q \)-exponential type-1 distribution \((0 < q < 1)\) if its pdf is given by
\[
f(x) = \nu(2-q)[1-(1-q)(\nu x)]^{1-q} \nu < 1, \quad 0 \leq x \leq \beta, \nu > 0 \tag{1.5}
\]
where \( \beta = \frac{1}{\nu(1-q)} \)

and the corresponding df is
\[
\overline{F}(x) = [1-(1-q)(\nu x)]^{\frac{2-q}{1-q}} \tag{1.6}
\]

Therefore, in view of (1.5) and (1.6), we have
\[
\overline{F}(x) = \frac{[1-(1-q)(\nu x)]^{\frac{2-q}{\nu(2-q)}}}{f(x)} \tag{1.7}
\]

Kamps (1998) investigated the importance of recurrence relations of order statistics in characterization. Recurrence relations for moments of order statistics and upper k-records were investigated, among others, by Khan et al. (1983a, 1983b), Grudzien and Szynal (1997) and Pawlas and Szynal (1998, 1999) among others.

In this paper, we are concerned with the generalized order statistics from \( q \)-exponential type-1 distribution. Sections 2 and 3 give the recurrence relations for single and product moments of generalized order statistics. Section 4 is based on the characterization result.
2. RECURRENCE RELATIONS FOR SINGLE MOMENTS

THEOREM 2.1: For the \( q \)-exponential type-1 distribution given (1.5) and \( n \in N, m \in R, 2 \leq r \leq n \)

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \\
\frac{j}{\gamma, \nu(2-q)} \left\{ E[X^{j-1}(r,n,m,k)] - \nu(1-q) [E[X^j(r,n,m,k)]] \right\}
\]

(2.1)

PROOF: From (1.3), we have

\[
E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\beta x^j[F(x)]^{\gamma-1} f(x) g_m^{-1}(F(x)) dx
\]

(2.2)

Integrating by parts taking \([\Gamma(x)]^{\gamma-1} f(x)\) as the part to be integrated, we get

\[
E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)] - \frac{jC_{r-1}}{\gamma, (r-1)!} \left\{ \int_0^\beta x^{j-1}[\Gamma(x)]^{\gamma} g_m^{-1}(F(x)) dx \right\}
\]

The constant of integration vanishes since the integral considered in (2.2) is a definite integral, on using (1.7), we obtain

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \frac{j}{\gamma, \nu(2-q)} \left\{ E[X^{j-1}(r,n,m,k)] - \nu(1-q) [E[X^j(r,n,m,k)]] \right\}
\]

and hence the Theorem

REMARK 2.1: Setting \( m = 0, k = 1 \) in the Theorem 2.1, we obtain the recurrence relations for the single moments of order statistics of the \( q \)-exponential type-1 distribution in the form

\[
E[X^j_{r,n}] - E[X^j_{r-1,n}] = \frac{j}{\nu(2-q)(n-r+1)} \left\{ E[X^{j-1}_{r,n}] - \nu(1-q) [E[X^j_{r,n}]] \right\}
\]

REMARK 2.2: Setting \( m = -1, k = 1 \) in the Theorem 2.1, we get the recurrence relations for the single moments of upper \( k \) record of the \( q \)-exponential type-1 distribution in the form

\[
E[X^j_{U(\nu)}]^k - E[X^j_{U(\nu-1)}]^k = \frac{j}{\nu(2-q)k} \left\{ E[X^{j-1}_{U(\nu)}]^k - \nu(1-q) [E[X^j_{U(\nu)}]^k] \right\}
\]

3. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

THEOREM 3.1: For the \( q \)-exponential type-1 distribution given (1.5) and \( n \in N, m \in R, 1 \leq r \leq s \leq n - 1 \)

\[
E[X^j(r,n,m,k) X^i(s,n,m,k)] - E[X^j(r,n,m,k) X^i(s-1,n,m,k)] = \\
\frac{j}{\gamma, \nu(2-q)} \left\{ E[X^{j-1}(r,n,m,k) X^{i-1}(s,n,m,k)] - \nu(1-q) [E[X^j(r,n,m,k) X^i(s,n,m,k)]] \right\}
\]

(3.1)

PROOF: From (1.4), we have

\[
E[X^j(r,n,m,k) X^i(s,n,m,k)] = \frac{C_{r-1}}{(r-1)!(s-r-1)!} \left\{ \int_0^\beta x^j[F(x)]^m f(x) g_m^{-1}(F(x)) \right\} I(x) dx
\]

(3.2)

where

\[
I(x) = \int_0^\beta y^i[F(x)]^{\gamma-1} \left[ h_m(F(x)) - h_m(F(x))^{s-r-1} f(y) \right] dy
\]
Solving the integral in \( I(x) \) by parts and substituting the resulting expression in (3.2), we get

\[
E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)] = \frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \int_0^\beta \int_0^{x_1} x^1 y^{r-1} \left[ \tilde{F}(x) \right]^{in_m} f(x) g_{n,m}^{r-1} (F(x) \\
\times [h_m(F(y) - h_m(F(x))]^{r-1} [F(y)]^{r-1} dy dx
\]

The constant of integration vanishes since the integral in \( I(x) \) is definite integral. On using relation (1.7), we obtain

\[
E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)] = \]

\[
\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \left[ \int_0^\beta \int_0^{x_1} x^1 y^{r-1} \left[ \tilde{F}(x) \right]^{in_m} f(x) g_{n,m}^{r-1} (F(x) \\
\times [h_m(F(y) - h_m(F(x))]^{r-1} [F(y)]^{r-1} f(y) dy dx
\right]
\]

\[
E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)] = \]

\[
\frac{j}{\gamma_s(r-1)!} E[X^i(r,n,m,k)X^j(s-1,n,m,k)] - \nu(1-q) E[X^i(r,n,m,k)X^j(s,n,m,k)]
\]

and hence the Theorem

**REMARK 3.1:** Setting \( m = 0, k = 1 \) in the Theorem 3.1, we obtain the recurrence relations for the product moments of order statistics of the \( q \) - exponential type-1 distribution in the form

\[
E[X_{r,s,n}^{ij}] - E[X_{r-1,s,n}^{ij}] = \frac{j}{\nu(2-q)(n-s+1)} \left[ E[X_{r,s,n}^{ij-1}] - \nu(1-q) E[X_{r,s,n}^{ij}] \right]
\]

**REMARK 3.2:** Setting \( m = -1, k = 1 \) in the Theorem 3.1, we get the recurrence relations for the product moments of upper \( k \) - record of the \( q \) - exponential type-1 distribution in the form

\[
E[X_{r,U(r)}^{ij} X_{r,U(r)}] - E[X_{r-1,U(r)}^{ij} X_{r-1,U(r)}] = \frac{j}{\nu(2-q)k} \left[ E[X_{r,U(r)}^{ij-1} X_{r,U(r)}] - \nu(1-q) E[X_{r,U(r)}^{ij} X_{r,U(r)}] \right]
\]

**4. CHARACTERIZATION**

**THEOREM 4.1:** Let \( X \) be a non-negative random variable having absolutely continuous distribution \( F(x) \) with \( F(0) = 0 \) and \( 0 < F(x) < 1 \), for all \( x > 0 \)

\[
E[X^i(r,n,m,k)] = E[X^i(r-1,n,m,k)] + \frac{j}{\gamma_r \nu(2-q)} E[X^{i-1}(r,n,m,k)] - \frac{j(1-q)}{\gamma_r(2-q)} E[X^i(r,n,m,k)]
\]

if and only if

\[
\tilde{F}(x) = [1 - (1-q)(\nu,x)]^{2-q}
\]
\textbf{Proof:} The necessary part follows immediately from equation (2.1). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equation (1.3), we have
\[
\frac{C_{r-1}}{(r-1)!} \int_0^\beta x^j [F(x)] f(x) g_{m-1}^{r-1} (F(x)) dx = \frac{(r-1)C_{r-1}}{\gamma_r (r-1)!} \int_0^\beta x^j [F(x)] f(x) g_{m-1}^{r+m} (F(x)) dx
\]
\[ + \frac{j C_{r-1}}{\gamma_r (r-1)!} \int_0^\beta x^{j-1} [F(x)] f(x) g_{m-1}^{r-1} (F(x)) dx \]
\[ - \frac{j C_{r-1} (1-q)}{\gamma_r (r-1)! (2-q)} \int_0^\beta x^{j-1} [F(x)] f(x) g_{m-1}^{r+1} (F(x)) dx \]
(4.2)

Integrating the first integral on the right hand side of equation (4.2), by parts, we get
\[
\frac{C_{r-1}}{(r-1)!} \int_0^\beta x^j [F(x)] f(x) g_{m-1}^{r-1} (F(x)) dx = - \frac{j C_{r-1}}{\gamma_r (r-1)!} \int_0^\beta x^{j-1} [F(x)] f(x) g_{m-1}^{r-1} (F(x)) dx
\]
\[ + \frac{j C_{r-1}}{\gamma_r (r-1)! (2-q)} \int_0^\beta x^{j-1} [F(x)] f(x) g_{m-1}^{r-1} (F(x)) dx \]
\[ + \frac{j C_{r-1}}{\gamma_r (r-1)! (2-q)} \int_0^\beta x^{j-1} [F(x)] f(x) g_{m-1}^{r+1} (F(x)) dx \]
which reduces to
\[
\frac{j C_{r-1}}{\gamma_r (r-1)!} \int_0^\beta x^{j-1} [F(x)] g_{m-1}^{r-1} (F(x)) [F(x) - \frac{1}{v(2-q)} f(x) + \frac{(1-q)}{(2-q)} f(x)] dx = 0 .
\]
(4.3)

Now applying a generalization of the Muntz- Szasz Theorem (Hawang and Lin, 1984) to equation (4.3), we get
\[
\frac{f(x)}{\mathcal{F}(x)} = \frac{v(2-q)}{[1-(1-q)v x]^{2-q}}
\]
which proves that
\[
\mathcal{F}(x) = [1-(1-q)v x]^{1-q}
\]

\textbf{ACKNOWLEDGMENT}

The authors acknowledge the learned referee for his comments and suggestions that substantially improved this present manuscript.

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