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Some Bivariate Gamma Distributions

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Abstract: In this paper, we find integral presentations of incomplete gamma functions. Using these results we construct bivariate gamma distributions and prove some properties of such distributions.

Keywords: Incomplete gamma functions, Bivariate gamma distributions.

1. INTRODUCTION

Definition 1.1: For *Re(s)>0*, the lower incomplete gamma function is defined as:

$$\gamma(s,x) = \int_0^x t^{s-1}\ e^{-t}dt\,,$$

and the upper incomplete gamma function is defined as

$$\Gamma(s,x) = \int_x^\infty t^{s-1}\ e^{-t}dt\,.$$

Clearly,

$$\Gamma(s,x) = \gamma(s) - \gamma(s,x) \qquad \textbf{(1.1)}$$

Moreover, $\gamma(s,x) \longrightarrow \gamma(s)$ as $x \longrightarrow \infty$ and $\Gamma(s,0) = \Gamma(s)$.

Definition 1.2: The beta function is defined as follows: *for* $0 \le x \le 1, Re(s)>0$ and $Re(t)>0$

$$\beta(s,t) = \int_0^x u^{s-1}(1-u)^{t-1}\ dt\,.$$

The properties of these functions are listed in many references (for example see[5], [7], and [12]). In particular, the following properties are needed:

Proposition 1.3: (see **[3]**) For *Re(s)>0*

1) $\Gamma(1,x) = e^{-x}$,
2) $\Gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}\,\mathrm{erfc}(\sqrt{x})$,
3) $\gamma(1,x) = 1 - e^{-x}$,
4) $\gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}\,\mathrm{erf}(\sqrt{x})$,

The incomplete gamma functions are used in the discussions of power-law relaxation times in complex physical systems (Sornette ([11]); logarithmic oscillations in relaxation times for proteins (Metzler et al. ([6])); Gaussian orbitals and exponential (Slater) orbitals in quantum chemistry (Shavitt ([9]), Shavitt and Karplus ([10])).
Recently, products of Incomplete gamma function are presented as integrals in [1] and [2].

Proposition 1.4: For $k \neq 0$

$$\int_0^t r^s\ e^{-kr}dr = \frac{1}{k^{s+1}}\gamma(s+1,kt).$$

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PROOF: The substitution $u=kr$ gives

$$\int_0^t r^s\ e^{-kr}dr=\frac{1}{k^{s+1}}\int_0^{kt}u^s\ e^{-u}du=\frac{1}{k^{s+1}}\gamma(s+1,kt).$$

As a remark the equation is valid for the case when $k=0$, in this case L'Hòpital's rule is needed.

Remark 1.5: The transforms $u=zw$ and $v=z(1-w)$ maps the region

$$\Omega_1=\{(u,v): 0<u<t, 0<v<\infty\}$$

into

$$\Omega_2=\{(w,z): 0<w<1, 0<z<\frac{t}{w}\}.$$

2. BIVARIATE GAMMA DISTRIBUTIONS

A bivariate gamma distribution constructed from specified gamma marginals may be useful for representing joint probabilistic properties of multivariate hydrological events such as floods and storms (see [14]).

In this section, we construct bivariate gamma distributions and prove some properties of such distributions. The basis for their construction is the following characterization of gamma and beta distributions due to Yeo and Milne [13]. Recent bivariate gamma is constructed in [8].

Definition 2.1: A random variable X is beta distributed with shape parameters s and t if its probability density function is

$$f(x)=\frac{x^{s-1}(1-x)^{t-1}}{\beta(s,t)}, 0<x<1; s>0, t>0.$$

Definition 2.2: A random variable X is gamma distributed with shape parameters s and scale parameter t if its probability density function is

$$f(x)=\frac{t^s x^{s-1}exp(-tx)}{\Gamma(s)}, x>0; s>0, t>0.$$

Assume that W is beta distributed with shape parameters b_1 and b_2 . Assume further that U and V are gamma distributed with shape parameters a_1 and a_2 respectively, and the scale parameters $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$, respectively. The question here

What is the joint probability density function of $X = U W$ and $Y = V W$?

An answer is given in [8] for the case when $a_1=a_2=b_1+b_2$.

In this paper, we give an answer for the case $b_1+b_2-a_1-a_2=1$.

Now, we prove the following result which is needed later

Theorem 2.3: For $t>0$, $Re(a)>0$ and $Re(b)>0$. The product $\gamma(a,t)\Gamma(b)$ satisfies

$$\gamma(a,t)\Gamma(b)=\int_0^1\gamma\left(a+b,\frac{t}{w}\right)w^{a-1}(1-w)^{b-1}\ dw.$$

PROOF: The transforms $u=zw$ and $v=z(1-w)$, Proposition 1.4, Remark 2.3, and Definition 1.1. give

$$\gamma(a,t)\Gamma(b)=\int_0^t\int_0^\infty e^{-u}u^{a-1}\ e^{-v}v^{b-1}dv\ du$$
$$=\int_0^1\int_0^{\frac{t}{w}} e^{-zw}(zw)^{a-1}\ e^{-z(1-w)}(z(1-w))^{b-1}(zdz\ dw)$$
$$=\int_0^1\gamma\left(a+b,\frac{t}{w}\right)w^{a-1}(1-w)^{b-1}\ dw.$$

Equation 1.1. and Proposition 1.3 give

Corollary 2.4 : For $t>0$, $Re(a)>0$ and $Re(b)>0$. The product $\Gamma(a,t)\Gamma(b)$satisfies

$$\Gamma(a,t)\Gamma(b)=\int_0^1\Gamma\left(a+b,\frac{t}{w}\right)w^{a-1}(1-w)^{b-1}\ dw.$$

In particular, for $0<Re(a)<1$ and $t>0$

$$\Gamma(a,t)\Gamma(1-a)=\int_0^1 \frac{e^{-\frac{t}{w}}}{(1-w)^a w^{1-a}}\,dw.$$

Theorem 2.4: Assume that *W, U* and *V are independent random variables where W* is beta distributed with shape parameters b_1 and b_2, *U* and *V* are gamma distributed with shape parameters a_1 and a_2 and scale parameters $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$, respectively where $b_1+b_2-a_1-a_2=1$. Define $X = U W$ and $Y = V W$**,** then the joint distribution of *X* and *Y* is given as

$$f(x,y)=\frac{x^{a_1-1}y^{a_2-1}\Gamma(b_1-a_1-a_2,\frac{x}{\mu_1}+\frac{y}{\mu_2})}{\Gamma(a_1)\mu_1{}^{a_1}\mu_2{}^{a_2}\beta(b_1,b_2)}$$

PROOF:

$$f(u,v,w)=\frac{\frac{u^{a_1-1}}{\mu_1{}^{a_1}}}{\Gamma(a_1)}\,exp\left(-\frac{u}{\mu_1}\right)\frac{\frac{u^{a_1-1}}{\mu_1{}^{a_1}}}{\Gamma(a_1)}\,exp\left(-\frac{v}{\mu_2}\right)\;\frac{w^{b_1-1}(1-w)^{b_2-1}}{\beta(b_1,b_2)}$$
$$=\frac{u^{a_1-1}u^{a_2-1}w^{b_1-1}(1-w)^{b_2-1}}{\mu_1{}^{a_1}\mu_1{}^{a_1}\Gamma(a_1)\Gamma(a_1)\beta(b_1,b_2)}\;exp\left(-(\frac{u}{\mu_1}+\frac{v}{\mu_2})\right).$$

Therefore,

$$f(x,y,w)=\frac{x^{a_1-1}y^{a_2-1}w^{b_1-1-a_1-a_2}(1-w)^{b_2-1}}{\mu_1{}^{a_1}\mu_1{}^{a_1}\Gamma(a_1)\Gamma(a_1)\beta(b_1,b_2)}\;exp\left(-\frac{1}{w}\left(\frac{u}{\mu_1}+\frac{v}{\mu_2}\right)\right).$$

Then

$$f(x,y)=\frac{x^{a_1-1}y^{a_2-1}}{\mu_1{}^{a_1}\mu_1{}^{a_1}\Gamma(a_1)\Gamma(a_1)\beta(b_1,b_2)}\int_0^1 w^{b_1-1-a_1-a_2}(1-w)^{b_2-1}exp\left(-\frac{1}{w}\left(\frac{u}{\mu_1}+\frac{v}{\mu_2}\right)\right)dw\;.$$

If $b_1+b_2-a_1-a_2=1$ and using Equation 2.4, we get

$$f(x,y)=\frac{x^{a_1-1}y^{a_2-1}\Gamma(b_1-a_1-a_2,\frac{x}{\mu_1}+\frac{y}{\mu_2})}{\Gamma(a_1)\mu_1{}^{a_1}\mu_2{}^{a_2}\beta(b_1,b_2)}.$$

The following are the contour diagram and 3D representation of this distribution for the case when $b_1=4.5$, $b_2=0.5$, $a_1=2.5$ and $a_2=1.5$ and the contour diagram and 3D representation of the distribution for the case constructed when a=1, b=1, and c = 2. The densities of bivariate gamma distribution in this paper and th e one constructed in [8] look different.

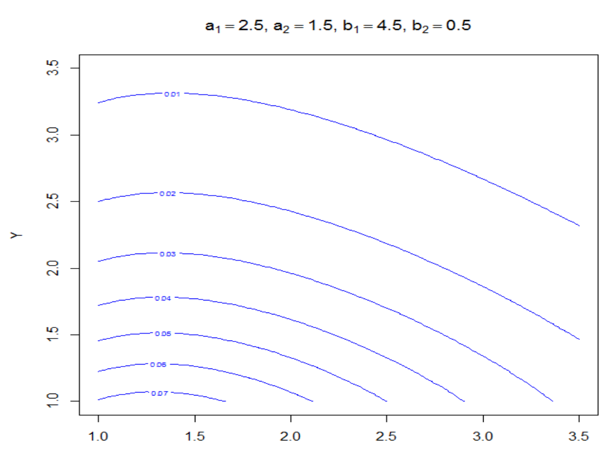

Figure 1. Contour diagram of f(x, y)

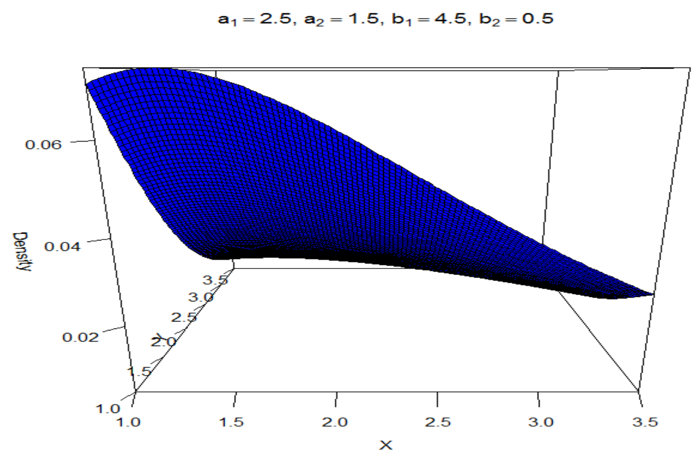

Figure 2. 3D representation of $f(x,y)$

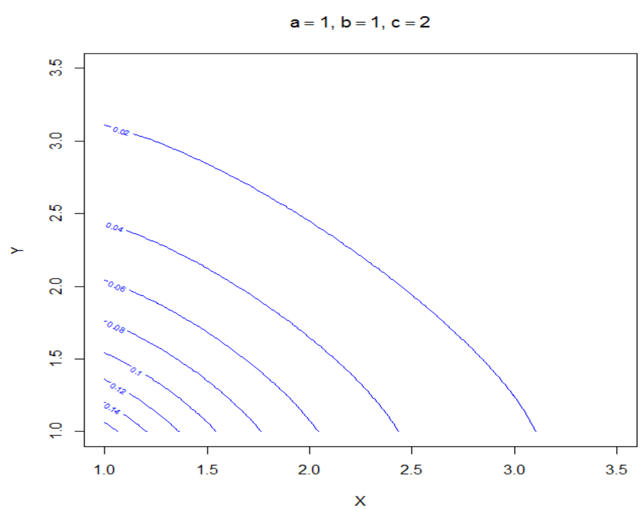

Figure 3. Contour diagram of $f(x,y)$ in [8]

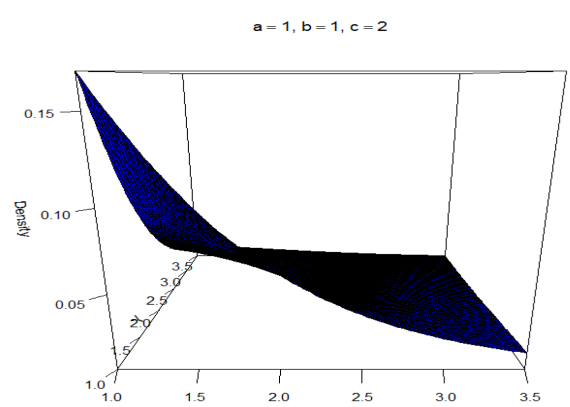

Figure 4. 3D representation of $f(x,y)$ in [8]

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