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Characterization of a New Class of Generalized Pearson Distribution by Truncated Moment

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Abstract: A probability distribution can be characterized through various methods. In this paper, by using truncated moment, we present some characterizations of a new class of generalized Pearson distribution introduced by Shakil and Singh [12].

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1. Introduction

A probability distribution can be characterized through various methods, see, for example, Ahsanullah et al. [1], among others. Since the characterizations of probability distributions play an important part in the determination of distributions by using certain properties in the given data, there has been a great interest, in recent years, in the characterizations of probability distributions by truncated moments. For example, the development of the general theory of the characterizations of probability distributions by truncated moment began with the work of Galambos and Kotz [5]. Further development continued with the contributions of many authors and researchers, among them Kotz and Shanbhag [11], Glänzel et al. [7], and Glänzel [6] are notable. Most of these characterizations are based on a simple relationship between two different moments truncated from the left at the same point. As pointed out by Glänzel [6], these characterizations may serve as a basis for parameter estimation. The characterizations by truncated moments may also be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions. For example, as pointed out by Kim and Jeon [10], in actuarial science, the credibility theory proposed by Buhlmann [4] allows actuaries to estimate the conditional mean loss for a given risk to establish an adequate premium to cover the insured's loss. In their paper, Kim and Jeon [10] have proposed a credibility theory based on the truncation of the loss data, or the trimmed mean, which also contains the classical credibility theory of Buhlmann [4] as a special case. It appears from the literature that not much attention has been paid to the characterization of the said new class of generalized Pearson distribution distribution introduced by Shakil and Singh [12], except the papers of Bondesson [2] and Hamedani [9], in which they have provided the characterizations of certain families of generalized gamma convolution (GGC) distributions, known as $\Im - class$ of distributions in honor of Professor O. Thorin, cf. Bondesson [3]. We would like to point out that the class of distributions introduced by Shakil and Singh [12] is quite different from the $\Im - class$ of distributions introduced and characterized by Bondesson [2] and later by Hamedani [9]. To see the difference, we refer the interested reader to the respective papers of Bondesson [2], Hamedani [9], and Shakil and Singh [12]. In this paper, we have established a characterization by truncated first moment of a new class of generalized Pearson distribution introduced by Shakil and Singh [12].

The paper is organized as follows. Shakil-Singh distribution [12] and some of its properties are discussed in Section 2. We present our characterization results in Section 3. Concluding remarks are presented in Section 4.

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2. The New Class of Generalized Pearson Distribution

For a positive continuous random variable X, Shakil and Singh [12] define a new class of univariate continuous distributions, which we shall call hereafter as Shakil-Singh distribution, based on the following generalized Pearson differential equation

$$\frac{1}{f(x)}\frac{df\,(x)}{dx} = -\frac{a_1 + b_1 x}{a_2 + b_2 x}\,, \tag{1}$$

where $m = 1,\, n = 1$, $a_2 = a_3 = \cdots = a_m = 0$, $b_2 = b_3 = \cdots = b_n = 0$, and where the right-hand expression, $\frac{a_1 + b_1 x}{a_2 + b_2 x}$, i.e., the ratio of two polynomials of first degree in x, with $a_1, b_1 \geq 0;\, a_2, b_2 > 0;\, a_1 b_2 \neq a_2 b_1$ (except when $a_1, b_1 = 0$), is known as Michaelis-Menten function. By solving the differential equation (1), the probability density function (pdf) of Shakil-Singh distribution is given by

$$f_X(x) = C\,(1 + \alpha\, x)^{\mu}\,(\gamma + \delta\, x)^{-\theta}\, e^{-\lambda x},\ \ \alpha, \gamma, \delta > 0;\ \mu, \theta, \lambda \geq 0;\ x > 0\,, \tag{2}$$

where $\mu = \frac{a_2\, b_1}{{b_2}^2}$, $\alpha = \frac{b_2}{a_2}$, $\gamma = a_2$, $\delta = b_2$, $\theta = \frac{a_1}{b_2}$, $\lambda = \frac{b_1}{b_2}$, $(a_2, b_2 \neq 0)$, and C denotes the normalizing constant given by

$$C = \left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k\,(-1)^j\,(\theta)_k\,(-\mu)_j\,\gamma^{-\theta}\left(\frac{\delta}{\gamma}\right)^k \alpha^j\,\lambda^{-k-j-1}\,(k+j)!}{k!\,j!}\right]^{-1}, \tag{3}$$

or

$$C = \left[\sum_{j=0}^{\infty}\delta^{-\theta}\,\alpha^j\,(\mu)_j\left(\frac{\gamma}{\delta}\right)^{j+1-\theta}\Psi\left(j+1,\ j+2-\theta;\ \frac{\lambda\gamma}{\delta}\right)\right]^{-1}, \tag{4}$$

where $(s)_k = \frac{\Gamma(s+k)}{\Gamma(s)} = s(s+1)\ldots(s+k-1)$, $(s \neq 0)$, and $(s)_0 = 1$, denote the Pochhammer symbol, and $\Psi(p, q; z) = \frac{1}{\Gamma(p)}\int_0^{\infty} e^{-zt}\, t^{p-1}\,(1+p)^{q-p-1}\, dt$ is known as Kummer's (or degenerate hypergeometric) function of the second kind. We will call (2) hereafter as Shakil-Singh distribution. The cumulative distribution function (cdf) of Shakil-Singh distribution is given as follows

$$F_X(x) = \int_0^x f_X(x)\, dx = C\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k\,(\theta)_k\,(\mu)_j\,\gamma^{-\theta}\left(\frac{\delta}{\gamma}\right)^k \alpha^j\,\lambda^{-k-j-1}\,\gamma\left(k+j+1, \lambda x\right)}{k!\,j!}\,, \tag{5}$$

where $\gamma(s, z) = \int_0^z t^{s-1} e^{-t}\, dt$ denotes the incomplete gamma function, and C denotes the normalizing constant given by (2). The nth moment is given by

$$E\left(X^n\right)=C\sum_{j=0}^{\infty}\frac{\delta^{-\theta}\,\alpha^{j}\,(\mu)_j\left(\frac{\delta}{\gamma}\right)^{n+j+1-\theta}\Gamma(n+j+1)\Psi\left(n+j+1,n+j+2-\theta;\frac{\lambda\delta}{\gamma}\right)}{j!}.\ (6)$$

As pointed out by Shakil and Singh [12], by a simple transformation of the variable X or by taking special values of the parameters $\{\alpha, \gamma, \delta > 0;\ \mu, \theta, \lambda \ge 0\}$, number distributions are special cases of their distribution as stated below.

i) Pearson III Distribution (when $\theta = 0$).

ii) Pearson VIII Distribution (when $\mu = 0$, $\lambda = 0$).

iii) Pearson IX Distribution (when $\lambda = 0$, $\theta = 0$).

iv) Pearson X Distribution (when $\mu = 0$, $\theta = 0$).

v) A Special Case of the pdf of the Shakil-Singh distribution (when $\mu = 0$) is given by

$$f_X(x) = C\,(\gamma+\delta x)^{-\theta}\,e^{-\lambda x},\ \alpha, \gamma, \delta > 0;\ \theta, \lambda \ge 0;\ x > 0,$$

where C denotes the normalizing constant given by

$$C=\left(\frac{\gamma}{\delta}\right)^{\theta}\left[\gamma\ \ \Psi\left(1, 2-\theta; \frac{\lambda\gamma}{\delta}\right)\right]^{-1}.$$

vi) It can easily be seen that, by a simple transformation of the variable X or by taking special values of the parameters $\{\alpha, \gamma, \delta > 0;\ \mu, \theta, \lambda \ge 0\}$, the pdf of the above special case (v) of our proposed distribution, when $\mu = 0$, can be expressed as the pdf of the product of the pdf's of the exponential and some members of the family of Burr distributions (such as Lomax, or Pareto Type I, or Pareto Type II distributions).

For a detailed treatment of the above distribution, we refer the interested reader to Shakil and Singh [12].

3. CHARACTERIZATIONS

For proving our main results, we will need some assumptions and lemmas which are provided in Subsection 3.1 as Assumptions 3.1.1, and Lemmas 3.1.1 and 3.1.2. We provide in Subsection 3.2, the proposed characterizations by truncated moment of the Shakil and Singh distribution in Theorems 3.2.1 and 3.2.2.

3.1. Assumption and Lemmas

Assumptions 3.1.1: Suppose that X be an absolutely continuous random variable with cumulative distribution function (cdf) $F(x)$ and the probability density function (pdf) $f(x)$. Let $\omega = \inf\{x \mid F(x) > 0\}$, and $\xi = \sup\{x \mid F(x) < 1\}$. We assume that $E(X)$ exists and $f(x)$ be a differentiable function for all x in (ω, ξ).

Lemma 3.1.1: Under the Assumptions 3.1.1, if $E(X \mid X \le x) = g(x)\eta(x)$, where $g(x)$ is a differentiable function for all x in (ω, ξ) and $\eta(x) = \dfrac{f(x)}{F(x)}$, then $f(x) = c\,e^{\int \frac{x - g'(x)}{g(x)}dx}$, where c is a constant determined by the condition $\int_{\omega}^{\xi} f(x)dx = 1$.

Proof: Suppose that $E(X|X \le x) = g(x)\eta(x)$. Then, since $E(X|X \le x) = \frac{\int_{\omega}^{x} u\, f(u)du}{F(x)}$ and $\eta(x) = \frac{f(x)}{F(x)}$, we have $g(x) = \frac{\int_{\omega}^{x} u\, f(u)du}{f(x)}$, that is, $\int_{\omega}^{x} u\, f(u)du = f(x)g(x)$.

Differentiating both sides of the above equation with respect to x, we obtain

$$x\, f(x) = f'(x)g(x) + f(x)g'(x).$$

From the above equation, we obtain

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}.$$

On integrating the above equation with respect to x, we have

$$f(x) = c\, e^{\int \frac{x - g'(x)}{g(x)} dx},$$

where c is obtained by the condition $\int_{\omega}^{\xi} f(x)dx = 1$. This completes the proof of Lemma 3.1.1.

Note: Since cdf $F(x)$ is absolutely continuous (with respect to Lebesgue measure), then by Radon-Nikodym Theorem the pdf $f(x)$ exists and hence $\int_{-\infty}^{x} \frac{u - g'(u)}{g(u)} du$ exists. Also, note that in the above-stated Lemma 3.1.1, the left truncated conditional expectation of X considers a product of reverse hazard rate and another function of the truncated point.

Lemma 3.1.2: Under the Assumptions 3.1.1, if $E(X \mid X \ge x) = h(x)r(x)$, where $h(x)$ is a differentiable function for all x in (ω, ξ) and $r(x) = \frac{f(x)}{1 - F(x)}$, then $f(x) = c\, e^{-\int \frac{x + h'(x)}{h(x)} dx}$, where c is a constant determined by the condition $\int_{\omega}^{\xi} f(x)dx = 1$.

Proof: Suppose that $E(X \mid X \ge x) = h(x)r(x)$. Then, since $E(X \mid X \ge x) = \frac{\int_{x}^{\xi} u\, f(u)du}{1 - F(x)}$ and $r(x) = \frac{f(x)}{1 - F(x)}$, we have $h(x) = \frac{\int_{x}^{\xi} u\, f(u)du}{f(x)}$, that is, $\int_{x}^{\xi} u\, f(u)du = f(x)h(x)$.

Differentiating the above equation with respect to respect to x, we obtain

$$-x\, f(x) = f'(x)h(x) + f(x)\, h'(x).$$

From the above equation, we obtain

$$\frac{f'(x)}{f(x)} = -\frac{x+h'(x)}{h(x)}.$$

On integrating the above equation with respect to x, we have

$$f(x) = c\, e^{-\int \frac{x+h'(x)}{h(x)}dx},$$

where c is obtained by the condition $\int_{\omega}^{\xi} f(x)dx = 1$. This completes the proof of Lemma 3.1.2.

3.2. Main Characterization Results

In this sub-section, we provide in Theorems 3.2.1 and 3.2.2, the proposed characterizations by truncated moment of Shakil and Singh distribution, with the pdf (2) and the cdf (5).

Theorem 3.2.1: If the random variable X satisfies the Assumptions 3.1.1, with $\omega = 0$ and $\xi = \infty$, then $E(X|X \le x) = g(x)\eta(x)$, where $\eta(x) = \frac{f(x)}{F(x)}$, and

$$g(x) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k(-1)^j(\theta)_k(-\mu)_j \gamma^{-\theta}\left(\frac{\delta}{\gamma}\right)^k \alpha^j \lambda^{-j-k-2}\gamma(j+k+2,\lambda x)}{k!\,j!}}{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k(-1)^j(\theta)_k(-\mu)_j \gamma^{-\theta}\left(\frac{\delta}{\gamma}\right)^k \alpha^j e^{-\lambda x}}{k!\,j!}}, \qquad (7)$$

where $(s)_k = \frac{\Gamma(s+k)}{\Gamma(s)} = s(s+1)\ldots(s+k-1), (s \ne 0)$, and $(s)_0 = 1$, and $\gamma(s,z) = \int_0^z t^{s-1}e^{-t}dt,\ s > 0$ denotes the incomplete gamma function, if and only if X has the distribution with the pdf (2).

Proof: Suppose that

$$f_X(x) = \left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k(-1)^j(\theta)_k(-\mu)_j \gamma^{-\theta}\left(\frac{\delta}{\gamma}\right)^k \alpha^j \lambda^{-k-j-1}(k+j)!}{k!\,j!}\right]^{-1} (1+\alpha x)^{\mu}(\gamma+\delta x)^{-\theta} e^{-\lambda x},$$

$$\alpha, \gamma, \delta > 0;\ \mu, \theta, \lambda \ge 0;\ x > 0.$$

The, applying Lemma 3.1.1, using twice the binomial series representation $(1+w)^{-s} = \sum_{k=0}^{\infty}\frac{(s)_k(-w)^k}{k!}$, for any real value of s, and Eq. 3.381.3/ P. 317 of Gradshteyn and Ryzhik [8], where $(s)_k = \frac{\Gamma(s+k)}{\Gamma(s)} = s(s+1)\ldots(s+k-1), (s \ne 0)$, and $(s)_0 = 1$, denote the Pochhammer symbol, and integrating and simplifying, we easily have

$$g(x)=\frac{\int_0^x u\,f(u)du}{f(x)}$$

$$=\frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k(-1)^j(\theta)_k(-\mu)_j\,\gamma^{-\theta}\left(\frac{\delta}{\gamma}\right)^k\alpha^j\lambda^{-j-k-2}\gamma(j+k+2,\lambda x)}{k!\,j!}}{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k(-1)^j(\theta)_k(-\mu)_j\,\gamma^{-\theta}\left(\frac{\delta}{\gamma}\right)^k\alpha^j e^{-\lambda x}}{k!\,j!}}.$$

Consequently, $E(X\,|\,X\le x)=g(x)\eta(x)$, where $\eta(x)=\frac{f(x)}{F(x)}$, and, hence the proof of "if" part of the Theorem 3.2.1 follows from Lemma 3.1.1.

We will now prove the "only if" condition of the Theorem 3.2.1. Suppose that $E(X|X\le x)=g(x)\eta(x)=g(x)\,\frac{f(x)}{F(x)}$. Then, since $E(X|X\le x)=\frac{\int_{-\infty}^{x}u\,f(u)du}{F(x)}$, we have $g(x)=\frac{\int_0^x u\,f(u)du}{f(x)}$. Now, if the random variable X satisfies the Assumptions 3.1.1 and has the distribution, with the pdf (2), then we have

$$g(x)f(x)=\int_0^x u\,f(u)du \tag{8}$$

Differentiating both sides of (8) with respect to x, and applying the Fundamental Theorem of Calculus, it follows from Lemma 3.1.1 that

$$f'(x)g(x)+f(x)g'(x)=x\,f(x)$$

from which, we easily obtain

$$g'(x)=x-g(x)\left(\frac{\mu\alpha}{(1+\alpha x)}-\frac{\theta\delta}{(\gamma+\delta x)}-\lambda\right),$$

or,

$$\frac{x-g'(x)}{g(x)}=\frac{\mu\alpha}{(1+\alpha x)}-\frac{\theta\delta}{(\gamma+\delta x)}-\lambda.$$

Hence, by Lemma 3.1.1, we have

$$\frac{f'(x)}{f(x)}=\frac{\mu\alpha}{(1+\alpha x)}-\frac{\theta\delta}{(\gamma+\delta x)}-\lambda.$$

Integrating both sides of the above equation with respect to x, we obtain

$$f(x) = c\, e^{\int\left[\frac{\mu\alpha}{(1+\alpha x)} - \frac{\theta\delta}{(\gamma+\delta x)} - \lambda\right]dx}$$
$$= c(1+\alpha x)^{\mu}(\gamma+\delta x)^{-\theta} e^{-\lambda x}, \qquad (9)$$

where the constant c is determined by the relation $\int_0^\infty f(x)dx = 1$. Thus, in (9), using twice the binomial series representation $(1+w)^s = \sum_{k=0}^{\infty} \frac{(-1)^k (s)_k w^k}{k!}$, for any real s, and Eq. 3.351.3/ P. 310 of Gradshteyn and Ryzhik [8], where $(s)_k = \frac{\Gamma(s+k)}{\Gamma(s)} = s(s+1)\ldots(s+k-1), (s \neq 0)$, denotes the Pochhammer symbol, and then integrating both sides of the Eq. (9) with respect to x from 0 to $+\infty$, the normalizing constant c is easily given by

$$c = \left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k (-1)^j (\theta)_k (-\mu)_j \gamma^{-\theta} \left(\frac{\delta}{\gamma}\right)^k \alpha^j \lambda^{-k-j-1} (k+j)!}{k!\, j!}\right]^{-1},$$

where $\alpha, \gamma, \delta > 0; \mu, \theta, \lambda \geq 0$. This completes the proof of Theorem 3.2.1.

Theorem 3.2.2: If the random variable X satisfies the Assumptions 3.1.1, with $\omega = 0$ and $\xi = \infty$, then $E(X|X \geq x) = h(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$, and

$$h(x) = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k (-1)^j (\theta)_k (-\mu)_j \gamma^{-\theta} \left(\frac{\delta}{\gamma}\right)^k \alpha^j \lambda^{-j-k-2}\, \Gamma(j+k+2, \lambda x)}{k!\, j!}}{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k (-1)^j (\theta)_k (-\mu)_j \gamma^{-\theta} \left(\frac{\delta}{\gamma}\right)^k \alpha^j e^{-\lambda x}}{k!\, j!}}, \qquad (10)$$

where $(s)_k = \frac{\Gamma(s+k)}{\Gamma(s)} = s(s+1)\ldots(s+k-1), (s \neq 0)$, and $(s)_0 = 1$, and $\Gamma(s,z) = \int_z^\infty t^{s-1} e^{-t} dt$ denotes the (upper) incomplete gamma function, if and only if X has the distribution with the pdf (2).

Proof: Suppose that $E(X|X \geq x) = h(x)r(x) = h(x)\frac{f(x)}{1-F(x)}$. Then, since $E(X|X \geq x) = \frac{\int_x^\infty u f(u)du}{1-F(x)}$ and $r(x) = \frac{f(x)}{1-F(x)}$, we have $h(x) = \frac{\int_x^\infty u f(u)du}{f(x)}$. Now, if the random variable X satisfies the Assumptions 3.1.1 and has the distribution, with the pdf (2), then, after simplification, as shown in Theorem 3.2.1, we easily have

$$h(x) = \frac{\int_x^\infty u\, f(u) du}{f(x)}$$

$$= \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k (-1)^j (\theta)_k (-\mu)_j \gamma^{-\theta} \left(\frac{\delta}{\gamma}\right)^k \alpha^j \lambda^{-j-k-2} \Gamma(j+k+2, \lambda x)}{k!\, j!}}{\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k (-1)^j (\theta)_k (-\mu)_j \gamma^{-\theta} \left(\frac{\delta}{\gamma}\right)^k \alpha^j e^{-\lambda x}}{k!\, j!}} .$$

Consequently, $E(X | X \ge x) = h(x) r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$, and, hence, the proof of "if" part of the Theorem 3.2.2 follows from Lemma 3.1.2.

We will now prove the "only if" condition of the Theorem 3.2.2. Suppose that $E(X | X \ge x) = h(x) r(x) = h(x) \frac{f(x)}{1 - F(x)}$. Then, since $E(X | X \ge x) = \frac{\int_x^\infty u\, f(u) du}{1 - F(x)}$ and $r(x) = \frac{f(x)}{1 - F(x)}$, we have $h(x) = \frac{\int_x^\infty u\, f(u) du}{f(x)}$. Now, if the random variable X satisfies the Assumptions 3.1.1 and has the distribution, with the pdf (2), then we have

$$h(x) f(x) = \int_x^\infty u\, f(u) du . \tag{11}$$

Differentiating both sides of (11) with respect to x, it follows from Lemma 3.1.2 that

$$f'(x) h(x) + f(x) h'(x) = -x\, f(x) ,$$

from which, we easily obtain

$$h'(x) = -x - h(x) \left(\frac{\mu\alpha}{(1 + \alpha x)} - \frac{\theta\delta}{(\gamma + \delta x)} - \lambda \right),$$

or,

$$\frac{x + h'(x)}{h(x)} = -\frac{\mu\alpha}{(1 + \alpha x)} + \frac{\theta\delta}{(\gamma + \delta x)} + \lambda .$$

Hence, by Lemma 3.1.2, we have

$$\frac{f'(x)}{f(x)} = \frac{\mu\alpha}{(1 + \alpha x)} - \frac{\theta\delta}{(\gamma + \delta x)} - \lambda .$$

Integrating both sides of the above equation with respect to x, we obtain

$$f(x) = c\, e^{\int \left[\frac{\mu\alpha}{(1 + \alpha x)} - \frac{\theta\delta}{(\gamma + \delta x)} - \lambda \right] dx}$$

$$= c\,(1+\alpha x)^{\mu}\,(\gamma+\delta x)^{-\theta}\,e^{-\lambda x}, \tag{12}$$

where the constant c is determined by the relation $\int_0^\infty f(x)dx = 1$. Thus, integrating both sides of (12) with respect to x from 0 to $+\infty$, as shown in Theorem 3.2.1, we easily have

$$c = \left[\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^k (-1)^j (\theta)_k (-\mu)_j \gamma^{-\theta} \left(\frac{\delta}{\gamma}\right)^k \alpha^j \lambda^{-k-j-1} (k+j)!}{k!\, j!} \right]^{-1},$$

where $\alpha, \gamma, \delta > 0; \mu, \theta, \lambda \geq 0$. This completes the proof of Theorem 3.2.2.

4. CONCLUDING REMARKS AND SOME RECOMMENDATIONS

In this paper, we have considered a new class of generalized Pearson distribution introduced by Shakil and Singh [12], and provided its characterization by truncated moment method. We hope the findings of our paper may be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions. As pointed out by Glänzel [6], these characterizations may serve as a basis for parameter estimation. For example, in actuarial science, the credibility theory proposed by Buhlmann [4] allows actuaries to estimate the conditional mean loss for a given risk to establish an adequate premium to cover the insured's loss. In their paper, Kim and Jeon [10] have proposed a credibility theory based on the truncation of the loss data, or the trimmed mean, which also contains the classical credibility theory of Buhlmann [4] as a special case. Since the characterizations of probability distributions by truncated moments play an important part in the determination of distributions by using certain properties in the given data, it is hoped that the findings of our paper, combined with the proposed credibility theory of Kim and Jeon [10] based on truncation of the loss data, or the trimmed mean, may be useful for researchers in the fields of probability, statistics, and other applied sciences. In view of these assertions, the interested readers are strongly recommended to the papers of Buhlmann [4] and Kim and Jeon [10] for some numerical illustration based on the truncation of the loss data, or the trimmed mean, and how our proposed characterizations may be applied to these problems in credibility theory.

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DECLARATIONS: We confirm that none of the authors have any competing interests in the manuscript.

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