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On Concomitant of Order Statistics from Bivariate Log-Exponentiated Kumarswamy Distribution

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Abstract: In this article, the distribution of concomitant of order statistics has been obtained from Bivariate log-exponentiated Kumarswamy distribution. The distribution of the r^{th} order statistics and it's concomitant arising from Bivariate log-exponentiated Kumarswamy distribution has been deduced. The properties of concomitant arising from the corresponding order statistics are used to derive the results. The moment generating function (mgf) of concomitant of order statistics is also derived. We have also obtained the expression for the joint distribution of concomitants of two non-adjacent order statistics.

Keywords: Concomitant, Bivariate log-exponentiated Kumarswamy distribution, Order statistics.

1. INTRODUCTION

David (1973) introduced the concept of concomitant of order statistics as follows: Let $(X_i, Y_i);\ i=1,2,...,n$ be the n observations from a bivariate population. If we arrange X variates in some specific order then we get values of Y variate corresponding to each X values. These Y values are called concomitant of order statistics. If $X_{r:n};\ r=1,2,...,n$ be the r^{th} order statistics then r^{th} concomitant is denoted by $Y_{[r:n]};\ r=1,2,...,n$.

Application of concomitant of order statistics frequently arises in real life selection problem when the decision is based on two variables X and Y. e.g. X may be the score in preliminary test and Y may be the score in main test in any competitive test.

Elsherpieny *et al.* (2014) proposed the Bivariate log-exponentiated Kumarswamy distribution, denoted by $B\log-EK(\alpha_1,\alpha_2,\alpha_3,\lambda,\gamma)$, which has log-exponentiated Kumarswamy (log-EK) marginal. This distribution is obtained by using the method similar to that of method used by Kundu and Gupta (2009) to develop bivariate generalized exponential distribution. The properties of Blog-EK distribution are given below.
Univariate log-EK distribution with parameters α,λ,γ>0 has the probability density function (*pdf*) given as

$$f(x,\alpha,\lambda,\gamma)=\alpha\lambda\gamma e^{-x}(1-e^{-x})^{\gamma-1}[1-(1-e^{-x})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x})^{\gamma}\}\lambda]^{\alpha-1} \tag{1.1}$$

The cumulative distribution function (*cdf*) of univariate log-EK distribution is given as

$$F(x,\alpha,\lambda,\gamma)=[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha} \tag{1.2}$$

Let, $U_i \sim \log-EK(\alpha_i,\lambda,\gamma);\ i=1,2,3$ are independently distributed and consider the random variables

$$X=\max(U_1,U_2)$$
$$Y=\max(U_2,U_3)$$

Then $(X,Y)\sim B\log-EK(\alpha_1,\alpha_2,\alpha_3,\lambda,\gamma)$, where, $\alpha_1,\alpha_2,\alpha_3,\lambda,\gamma>0$, with *cdf* given as

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$$F(x,y)=P(U_1\le x)P(U_2\le y)P(U_3\le \min(x,y))$$
$$=F_{\log-EK}(x:\alpha_1,\lambda,\gamma)F_{\log-EK}(y:\alpha_2,\lambda,\gamma)F_{\log-EK}(z:\alpha_3,\lambda,\gamma) \tag{1.3}$$

Where $z=\min(x,y)$.

The joint *cdf* of the $B\log-EK(x,y:\alpha_1,\alpha_2,\alpha_3,\lambda,\gamma)$ given in (1.3) can also be written as

$$F(x,y)=\begin{cases}F_{\log-EK}(x:\alpha_1+\alpha_3,\lambda,\gamma)F_{\log-EK}(y:\alpha_2,\lambda,\gamma) & \text{if } 0<x<y\\ F_{\log-EK}(x:\alpha_1,\lambda,\gamma)F_{\log-EK}(y:\alpha_2+\alpha_3,\lambda,\gamma) & \text{if } 0<y<x\\ F_{\log-EK}(x:\alpha_1+\alpha_2+\alpha_3,\lambda,\gamma) & \text{if } y=x>0\end{cases}$$

And the corresponding *pdf* is

$$f(x,y)=\begin{cases}f_1(x,y) & \text{if } 0<x<y\\ f_2(x,y) & \text{if } 0<y<x\\ f_3(x,y) & \text{if } y=x>0\end{cases}$$

where

$$f_1(x,y)=\alpha_2(\alpha_1+\alpha_3)\lambda^2\gamma^2e^{-x}e^{-y}(1-e^{-x})^{\gamma-1}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-x})^{\gamma}\}^{\lambda-1}$$
$$\times\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2-1} \tag{1.4}$$

$$f_2(x,y)=\alpha_1(\alpha_2+\alpha_3)\lambda^2\gamma^2e^{-x}e^{-y}(1-e^{-x})^{\gamma-1}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-x})^{\gamma}\}^{\lambda-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}$$
$$\times[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2+\alpha_3-1} \tag{1.5}$$

$$f_3(x,y)=\alpha_3\lambda\gamma e^{-x}(1-e^{-x})^{\gamma-1}\{1-(1-e^{-x})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_2+\alpha_3-1} \tag{1.6}$$

The marginal *cdf* and *pdf* of X is

$$F(x)=[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3};0<x<\infty, \tag{1.7}$$

$$f(x)=(\alpha_1+\alpha_3)\lambda\gamma e^{-x}(1-e^{-x})^{\gamma-1}\{1-(1-e^{-x})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1};0<x<\infty \tag{1.8}$$

And the conditional *pdf* of Y given X is

$$f(y|x)=\begin{cases}f_1(y|x) & \text{if } 0<x<y\\ f_2(y|x) & \text{if } 0<y<x\\ f_3(y|x) & \text{if } y=x>0\end{cases}$$

where

$$f_1(y|x)=\alpha_2\lambda\gamma e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2-1} \tag{1.9}$$

$$f_2(y|x)=\frac{\alpha_1(\alpha_2+\alpha_3)}{(\alpha_1+\alpha_3)}\lambda\gamma e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2+\alpha_3-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{-\alpha_3} \tag{1.10}$$

$$f_3(y|x)=\frac{\alpha_3}{(\alpha_1+\alpha_3)}[1-\{1-(1-e^{-x})^{\lambda}\}^{\lambda}]^{\alpha_2} \tag{1.11}$$

Concomitant of order statistics was noticed and discussed about four decades ago. After remarkable work in 1973 by David; David and Galambos (1974) and David *et al.* (1977) deduced the asymptotic theory of concomitants of order statistics and distribution and expected value of the rank of a concomitant of an order statistics, respectively. Yang (1977) discussed the general distribution theory of the concomitants of order statistics. Bhattacharya (1984), in his paper, Induced order statistics: Theory and applications, explained the theory and application of concomitant of order statistics in the name of induced order statistics. Balasubramanian and Beg (1996) and (1997), obtained expressions of concomitant of order statistics in bivariate exponential distribution of Marshall and Olkin and concomitant of order statistics in Morgenstern type bivariate exponential distribution. Moreover, Begum and Khan (2000) and Begum (2003) obtained expressions of concomitant of order statistics from Marshall and Olkin's bivariate Weibull distribution and bivariate Pareto II distribution. In a similar manner, Shahbaz and Ahmad (2009) and Shahbaz *et al.* (2009) worked on concomitant of order statistics for bivariate Pseudo-Weibull Distribution and bivariate Pseudo Exponential Distribution. Very recently, Phillip and Thomas (2015) discussed the application of concomitant of order statistics from extended Farlie- Gumble- Morgenstern bivariate logistic distribution in estimation procedure.

In this paper, we have considered the *pdf* of concomitant of order statistics and the joint *pdf* of concomitant of order statistics when the pair of observations $(X_i, Y_i);\ i = 1, 2, ..., n$ are independent and follows $B\log\!-EK\left(\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma\right)$. The paper is divided into five sections. In section 2, we have obtained the distribution of r^{th} order statistics and joint distribution of r^{th} and s^{th} order statistics. In section 3, density of concomitant of first order statistics $Y_{[1:n]}$ and concomitant of r^{th} order statistics $Y_{[r:n]}$ have been deduced. In section 4, we have discussed the moment generating function of concomitant of r^{th} order statistics $Y_{[r:n]}$. Finally, in section 5, the expression for joint distribution of concomitant of two order statistics is obtained.

2. DISTRIBUTION OF ORDER STATISTICS

Let $X_1, X_2 ..., X_n$ are *iid* observations from a population having *cdf* $F(x)$ and *pdf* $f(x)$ then the *pdf* of r^{th} order statistics is given as

$$f_{r:n}(x) = C_{r:n}[1 - F(x)]^{n-r}[F(x)]^{r-1} f(x), \quad -\infty < x < \infty$$

where $C_{r:n} = \dfrac{n!}{(r-1)!(n-r)!}$.

Thus if $X_1, X_2 ..., X_n$ are *iid* observations from a population having *cdf* $F(x)$ and *pdf* $f(x)$ given in (1.7) and (1.8) respectively, then the *pdf* of r^{th} order statistics is given as

$$f_{r:n}(x) = C_{r:n}(\alpha_1 + \alpha_3)\lambda\gamma e^{-x}(1 - e^{-x})^{\gamma-1}\{1 - (1 - e^{-x})^{\gamma}\}^{\lambda-1} \times [1 - [1 - \{1 - (1 - e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-r}[1 - \{1 - (1 - e^{-x})^{\gamma}\}^{\lambda}]^{r(\alpha_1+\alpha_3)-1} \tag{2.1}$$

Putting $r = 1$ in (2.1), we get the *pdf* of first order statistics $X_{1:n}$ as

$$f_{1:n}(x) = n(\alpha_1 + \alpha_3)\lambda\gamma e^{-x}(1 - e^{-x})^{\gamma-1}\{1 - (1 - e^{-x})^{\gamma}\}^{\lambda-1} \times [1 - \{1 - (1 - e^{-x})^{\gamma}\}^{\lambda}]^{(\alpha_1+\alpha_3)-1}[1 - [1 - \{1 - (1 - e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-1} \tag{2.2}$$

Similarly the c*df* of r^{th} order statistics $X_{r:n}$, when $X_1, X_2 ..., X_n$ are *iid* observations and follows $\log\!-EK(\alpha_1 + \alpha_3, \gamma, \lambda)$, is given as

$$F_{r:n}(x) = [1 - (1 - [1 - \{1 - (1 - e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3})^{n-r+1}]\sum_{k=0}^{r-1}\binom{n-r+k}{k}[1 - \{1 - (1 - e^{-x})^{\gamma}\}^{\lambda}]^{k(\alpha_1+\alpha_3)} \tag{2.3}$$

The *cdf* of first order statistics $X_{1:n}$ as (David; 1981)

$$F_{1:n}(x) = 1 - [1 - [1 - \{1 - (1 - e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n} \tag{2.4}$$

Let $X_{r:n}$ and $X_{s:n}$ be the two order statistics such that $(1 \le r \le s < n)$. Then the joint *pdf* of two order statistics is given as (David; 1981)

$$f_{r:n}(x_1, x_2) = C_{r,s:n}[1 - \bar{F}(x_1)]^{r-1}[\bar{F}(x_1) - \bar{F}(x_2)]^{s-(r+1)}[\bar{F}(x_2)]^{n-s} f(x_1) f(x_2)$$

$$= C_{r,s:n} f(x_1) f(x_2) \sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{r-1}{l}\binom{s-r-1}{k}\left[\bar{F}(x_1)\right]^{l+s-r-k-1}\left[\bar{F}(x_2)\right]^{n-s+k}$$

Where $C_{r,s:n} = \dfrac{n!}{(r-1)![s-(r+1)]!(n-s)!}$ and $\bar{F}(x) = 1 - F(x).$

Therefore if $X_1, X_2 ..., X_n$ are *iid* observations from a population having *cdf* $F(x)$ and *pdf* $f(x)$ given in (1.7) and (1.8) respectively, then the joint *pdf* of $X_{r:n}$ and $X_{s:n}$ is given by

$$\begin{aligned}
f_{r,s:n}(x_1, x_2) &= C_{r,s:n}(\alpha_1+\alpha_3)^2\lambda^2\gamma^2 e^{-(x_1+x_2)}(1-e^{-x_1})^{\gamma-1}(1-e^{-x_2})^{\gamma-1}[1-(1-e^{-x_1})^{\gamma}]^{\lambda-1} \\
&\times[1-(1-e^{-x_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1}[1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1} \\
&\times\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{r-1}{l}\binom{s-r-1}{k}[1-(1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda})^{\alpha_1+\alpha_3}]^{n-s+k} \\
&\times[1-(1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda})^{\alpha_1+\alpha_3}]^{l+s-r-k-1}
\end{aligned} \tag{2.5}$$

3. DISTRIBUTION OF CONCOMITANT OF ORDER STATISTICS

In this section, we have deduced the result for the *pdf* of concomitant of order statistics. First we have obtained the *pdf* of concomitant of first order statistics $Y_{[1:n]}$ and utilizing *pdf* of $Y_{[1:n]}$, we have obtained the *pdf* of r^{th} concomitant of order statistics $Y_{[r:n]}$.

Let $(X_i, Y_i);\ i = 1, 2, ..., n$ are n pairs of observations and follows $B\log\text{–}EK(\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma)$, then the *pdf* of concomitant of first order statistics $Y_{[1:n]}$ is given by

$$g_{[1:n]}(y) = \int_0^y f_1(y\,|\,x) f_{1:n}(x)dx + \int_y^{\infty} f_2(y\,|\,x) f_{1:n}(x)dx + f_3(y\,|\,x) f_{1:n}(x)$$

Therefore in view of (1.9), (1.10), (1.11) and (2.2), we have

$$\begin{aligned}
g_{[1:n]}(y) &= n\alpha_2(\alpha_1+\alpha_3)\lambda^2\gamma^2 e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2-1} \\
&\times\int_0^y e^{-x}(1-e^{-x})^{\gamma-1}\{1-(1-e^{-x})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1}[1-[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-1}dx \\
&+n\alpha_1(\alpha_2+\alpha_3)\lambda^2\gamma^2 e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2+\alpha_3-1} \\
&\times\int_y^{\infty} e^{-x}(1-e^{-x})^{\gamma-1}\{1-(1-e^{-x})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1-1}[1-[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-1}dx \\
&+n\alpha_3\lambda\gamma e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_2+\alpha_3-1}[1-[1-\{1-(1-e^{-x})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-1}
\end{aligned}$$

After some algebraic simplification, we get

$$\begin{aligned}
&g_{[1:n]}(y)=\alpha_2\lambda\gamma e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2-1}[1-(1-[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3})^{n}] \\
&+n\alpha_1(\alpha_2+\alpha_3)\lambda\gamma e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2+\alpha_3-1} \\
&\times\sum_{q_0=0}^{n-1}\frac{(-1)^{q_0}}{[q_0(\alpha_1+\alpha_3)+\alpha_1]}\binom{n-1}{q_0}[1-(1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda})^{q_0(\alpha_1+\alpha_3)+\alpha_1}]+n\alpha_3\lambda\gamma e^{-y}(1-e^{-y})^{\gamma-1} \\
&\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_2+\alpha_3-1}[1-[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-1}
\end{aligned} \tag{3.1}$$

It is well known that *cdf* of order statistics are connected by the relation (see David; 1981)

$$F_{r:n}(x)=\sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}F_{1:i}(x);1\le r\le n$$

This relation also holds for *pdf* of concomitant of order statistics as (see Balasubramanian and Beg; 1998)

$$g_{[r:n]}(y)=\sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}g_{[1:i]}(y)$$

Therefore, the *pdf* of concomitant of r^{th} order statistics $Y_{[r:n]}$ is given as

$$\begin{aligned}
&g_{[r:n]}(y)=\sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\Big[\alpha_2\lambda\gamma e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2-1} \\
&\times[1-(1-[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3})^{i}]+i\alpha_1(\alpha_2+\alpha_3)\lambda\gamma e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1} \\
&\times[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_2+\alpha_3-1}\sum_{q_0=0}^{i-1}\frac{(-1)^{q_0}}{[q_0(\alpha_1+\alpha_3)+\alpha_1]}\binom{i-1}{q_0}[1-(1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda})^{q_0(\alpha_1+\alpha_3)+\alpha_1}] \\
&+i\alpha_3\lambda\gamma e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^{\gamma}\}^{\lambda-1}[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_2+\alpha_3-1}[1-[1-\{1-(1-e^{-y})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{i-1}\Big]
\end{aligned} \tag{3.2}$$

4. MOMENT GENERATING FUNCTION OF CONCOMITANT OF ORDER STATISTICS

Suppose $(X_i,Y_i);\ i=1,2,...,n$ are n pairs of observations from $B\log-EK(\alpha_1,\alpha_2,\alpha_3,\lambda,\gamma)$, then the moment generating function (mgf) of concomitant of first order statistics $Y_{[1:n]}$ is given by

$$M_{[1:n]}(t)=E[e^{tY_{[1:n]}}]=\int_0^{\infty}e^{ty}g_{[1:n]}(y)dy$$

Now in view of (3.1), we have

$$
\begin{aligned}
M_{[1:n]}(t) &= \alpha_2\lambda\gamma\int_0^\infty e^{ty}e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^\gamma\}^{\lambda-1}[1-\{1-(1-e^{-y})^\gamma\}^\lambda]^{\alpha_2-1}dy \\
&-\alpha_2\lambda\gamma\int_0^\infty e^{ty}e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^\gamma\}^{\lambda-1}[1-\{1-(1-e^{-y})^\gamma\}^\lambda]^{\alpha_2-1}[1-[1-\{1-(1-e^{-y})^\gamma\}^\lambda]^{\alpha_1+\alpha_3}]^n dy \\
&+n\alpha_1(\alpha_2+\alpha_3)\lambda\gamma\sum_{q_0=0}^{n-1}\frac{(-1)^{q_0}}{[q_0(\alpha_1+\alpha_3)+\alpha_1]}\binom{n-1}{q_0}\left[\int_0^\infty e^{ty}e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^\gamma\}^{\lambda-1}[1-\{1-(1-e^{-y})^\gamma\}^\lambda]^{\alpha_2+\alpha_3-1}dy\right. \\
&\left.-\int_0^\infty e^{ty}e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^\gamma\}^{\lambda-1}[1-\{1-(1-e^{-y})^\gamma\}^\lambda]^{(1+q_0)(\alpha_1+\alpha_3)+\alpha_2-1}dy\right] \\
&+n\alpha_3\lambda\gamma\int_0^\infty e^{ty}e^{-y}(1-e^{-y})^{\gamma-1}\{1-(1-e^{-y})^\gamma\}^{\lambda-1}[1-\{1-(1-e^{-y})^\gamma\}^\lambda]^{\alpha_1+\alpha_2+\alpha_3-1}[1-[1-\{1-(1-e^{-y})^\gamma\}^\lambda]^{\alpha_1+\alpha_3}]^{n-1}dy
\end{aligned}
\tag{4.1}
$$

Setting $1-(1-e^{-y})^\gamma = v,$ we have

$$
\begin{aligned}
M_{[1:n]}(t) &= \alpha_2\lambda\int_0^1\left(1-(1-v)^{\frac{1}{\gamma}}\right)^{-t}v^{\lambda-1}(1-v^\lambda)^{\alpha_2-1}dv-\alpha_2\lambda\int_0^1\left(1-(1-v)^{\frac{1}{\gamma}}\right)^{-t}v^{\lambda-1}(1-v^\lambda)^{\alpha_2-1}\left[1-(1-v^\lambda)^{\alpha_1+\alpha_3}\right]^n dv \\
&+n\alpha_1(\alpha_2+\alpha_3)\lambda\sum_{q_0=0}^{n-1}\frac{(-1)^{q_0}}{[q_0(\alpha_2+\alpha_3)+\alpha_1]}\binom{n-1}{q_0}\left[\int_0^1\left(1-(1-v)^{\frac{1}{\gamma}}\right)^{-t}v^{\lambda-1}(1-v^\lambda)^{\alpha_2+\alpha_3-1}dv\right. \\
&\left.-\int_0^1\left(1-(1-v)^{\frac{1}{\gamma}}\right)^{-t}v^{\lambda-1}(1-v^\lambda)^{(1+q_0)(\alpha_1+\alpha_3)+\alpha_2-1}dv\right]+n\alpha_3\lambda\int_0^1\left(1-(1-v)^{\frac{1}{\gamma}}\right)^{-t}v^{\lambda-1}(1-v^\lambda)^{\alpha_1+\alpha_2+\alpha_3-1}[1-(1-v^\lambda)^{\alpha_1+\alpha_3}]^{n-1}dv
\end{aligned}
\tag{4.2}
$$

Using the Taylor series expansion $[1-(1-v)^{\frac{1}{\gamma}}]^{-t}=\sum_{k_1=0}^\infty\frac{(t)_{K_1}}{k_1!}(1-v)^{\frac{k_1}{\gamma}}$ in (4.2), we get

$$
\begin{aligned}
M_{[1:n]}(t) &= \alpha_2\lambda\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty\frac{(t)_{k_1}(1-\alpha_2)_{k_2}}{k_1!k_2!}\int_0^1(1-v)^{\frac{k_1}{\gamma}}v^{\lambda(k_2+1)-1}dv \\
&-\alpha_2\lambda\sum_{q_1=0}^n\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty(-1)^{q_1}\binom{n}{q_1}\frac{(t)_{k_1}[1-q_1(\alpha_1+\alpha_3)-\alpha_2]_{k_2}}{k_1!k_2!}\int_0^1(1-v)^{\frac{k_1}{\gamma}}v^{\lambda(k_2+1)-1}dv \\
&+n\alpha_1(\alpha_2+\alpha_3)\lambda\sum_{q_0=0}^{n-1}\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty(-1)^{q_0}\binom{n-1}{q_0}\frac{(t)_{k_1}[1-\alpha_2-\alpha_3]_{k_2}}{[q_0(\alpha_1+\alpha_3)+\alpha_1]k_1!k_2!}\int_0^1(1-v)^{\frac{k_1}{\gamma}}v^{\lambda(k_2+1)-1}dv \\
&-n\alpha_1(\alpha_2+\alpha_3)\lambda\sum_{q_0=0}^{n-1}\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty(-1)^{q_0}\binom{n-1}{q_0}\frac{(t)_{k_1}[1-\alpha_2-(1+q_o)(\alpha_1+\alpha_3)]_{k_2}}{[q_0(\alpha_1+\alpha_3)+\alpha_1]k_1!k_2!}\int_0^1(1-v)^{\frac{k_1}{\gamma}}v^{\lambda(k_2+1)-1}dv \\
&+n\alpha_3\lambda\sum_{q_1=0}^{n-1}\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty(-1)^{q_1}\binom{n-1}{q_1}\frac{(t)_{k_1}[1-(\alpha_1+\alpha_3)(q_1+1)-\alpha_2]_{k_2}}{k_1!k_2!}\int_0^1(1-v)^{\frac{k_1}{\gamma}}v^{\lambda(k_2+1)-1}dv
\end{aligned}
$$

Using, $\int_0^1(1-v)^{\frac{k_1}{\gamma}}v^{\lambda(k_2+1)-1}dv=B\left[\lambda(k_2+1),\left(\frac{k_1}{\gamma}+1\right)\right]$

$$M_{[1:n]}(t)=\alpha_2\lambda\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(t)_{k_1}(1-\alpha_2)_{k_2}}{k_1!k_2!}B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right]-\alpha_2\lambda\sum_{q_1=0}^{n}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-1)^{q_1}\binom{n}{q_1}\frac{(t)_{k_1}[1-q_1(\alpha_1+\alpha_3)-\alpha_2]_{k_2}}{k_1!k_2!}B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right]$$
$$+n\alpha_1(\alpha_2+\alpha_3)\lambda\sum_{q_0=0}^{n-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-1)^{q_0}\binom{n-1}{q_0}\frac{(t)_{k_1}[1-\alpha_2-\alpha_3]_{k_2}}{[q_0(\alpha_1+\alpha_3)+\alpha_1]k_1!k_2!}B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right]-n\alpha_1(\alpha_2+\alpha_3)\lambda\sum_{q_0=0}^{n-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-1)^{q_0}\binom{n-1}{q_0}$$
$$\times\frac{(t)_{k_1}[1-\alpha_2-(1+q_0)(\alpha_1+\alpha_3)]_{k_2}}{[q_0(\alpha_1+\alpha_3)+\alpha_1]k_1!k_2!}B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right]+n\alpha_3\lambda\sum_{q_1=0}^{n-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-1)^{q_1}\binom{n-1}{q_1}\frac{(t)_{k_1}[1-\alpha_2-(1+q_1)(\alpha_1+\alpha_3)]_{k_2}}{k_1!k_2!}$$
$$B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right] \tag{4.3}$$

mgf of concomitant of r^{th} order statistics $Y_{[r:n]}$ can be obtained by using the relation (see Balasubramanian and Beg; 1998)

$$M_{[r:n]}(t)=\sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}M_{[1:n]}(t)$$

Therefore, in view of (4.3), we have

$$M_{[r:n]}(t)=\sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\Bigg[\alpha_2\lambda\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(t)_{k_1}(1-\alpha_2)_{k_2}}{k_1!k_2!}B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right]$$
$$-\alpha_2\lambda\sum_{q_1=0}^{i}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-1)^{q_1}\binom{i}{q_1}\frac{(t)_{k_1}[1-q_1(\alpha_1+\alpha_3)-\alpha_2]_{k_2}}{k_1!k_2!}B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right]$$
$$+i\alpha_1(\alpha_2+\alpha_3)\lambda\sum_{q_0=0}^{i-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-1)^{q_0}\binom{i-1}{q_0}\frac{(t)_{k_1}[1-\alpha_2-\alpha_3]_{k_2}}{[q_0(\alpha_1+\alpha_3)+\alpha_1]k_1!k_2!}B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right]$$
$$-i\alpha_1(\alpha_2+\alpha_3)\lambda\sum_{q_0=0}^{i-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-1)^{q_0}\binom{i-1}{q_0}\frac{(t)_{k_1}[1-\alpha_2-(1+q_0)(\alpha_1+\alpha_3)]_{k_2}}{[q_0(\alpha_1+\alpha_3)+\alpha_1]k_1!k_2!}B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right]$$
$$+i\alpha_3\lambda\sum_{q_1=0}^{i-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(-1)^{q_1}\binom{i-1}{q_1}\frac{(t)_{k_1}[1-(\alpha_1+\alpha_3)(q_1+1)-\alpha_2]_{k_2}}{k_1!k_2!}B\left[\lambda(k_2+1),\left(\frac{k_1}{\lambda}+1\right)\right]\Bigg]$$

as the required expression for mgf of concomitant of r^{th} order statistics $Y_{[r:n]}$.

5. JOINT DENSITY OF CONCOMITANT OF TWO ORDER STATISTICS

Let $Y_{[r:n]}$ and $Y_{[s:n]}$ be the concomitant of r^{th} and s^{th} order statistics, respectively. Then the joint *pdf* of $Y_{[r:n]}$ and $Y_{[s:n]}$ in general, is given by

$$g_{[r,s:n]}(y_1,y_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}f(y_1\mid x_1)f(y_2\mid x_2)f_{r,s:n}(x_1,x_2)dx_1dx_2$$

where $f_{r,s:n}(x_1,x_2)$ is the joint *pdf* of $X_{r:n}$ and $X_{s:n}$. Here, as we have *pdf* existing at three different conditions. So, the joint density of concomitant of two order statistics will be given as;

$$g_{[r,s:n]}(y_1,y_2)=I_1+I_2+I_3+I_4+I_5+I_6+I_7$$

where, $I_1,...,I_7$ are the joint densities under different conditions and are define below;

Case I: When $X<Y$

If: $0\le x_1<y_1<x_2<y_2\le\infty$; then

$$I_1=\int_{y_1}^{y_2}\int_0^{y_1} f_1(y_1\,|\,x_1)f_1(y_2\,|\,x_2)f_{r,s:n}(x_1,x_2)dx_1dx_2$$

Using (1.9) and (2.5), we have

$$\begin{aligned}I_1=&C_{r,s:n}(\alpha_1+\alpha_3)^2\lambda^4\gamma^4e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^{\gamma}]^{\lambda-1}[1-(1-e^{-y_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_2-1}\\&\times[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_2-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{s-r-1}{k}\binom{r-1}{l}\int_{y_1}^{y_2}e^{-x_2}(1-e^{-x_2})^{\gamma-1}[1-(1-e^{-x_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1}\\&\times[1-[1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-s+k}\{\int_0^{y_1}e^{-x_1}(1-e^{-x_1})^{\gamma-1}[1-(1-e^{-x_1})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1}\\&\times[1-[1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{l+s-r-k-1}dx_1\}dx_2\end{aligned}$$

on solving which we get

$$\begin{aligned}I_1=&C_{r,s:n}\lambda^2\gamma^2e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^{\gamma}]^{\lambda-1}[1-(1-e^{-y_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_2-1}\\&\times[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_2-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}\frac{(-1)^{l+k}}{(l+s-r-k)(n-s+k+1)}\binom{s-r-1}{k}\binom{r-1}{l}[1-[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{l+s-r-k}\\&\times[(1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3})^{n-s+k+1}-(1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3})^{n-s+k+1}]\end{aligned}$$

If: $0\le x_1<x_2<y_1<y_2\le\infty$; then

$$I_2=\int_0^{y_1}\int_0^{x_2} f_1(y_1\,|\,x_1)f_1(y_2\,|\,x_2)f_{r,s:n}(x_1,x_2)dx_1dx_2$$

Again using (1.9) and (2.5), we have

$$\begin{aligned}I_2=&C_{r,s:n}(\alpha_1+\alpha_3)^2\lambda^4\gamma^4e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^{\gamma}]^{\lambda-1}[1-(1-e^{-y_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_2-1}\\&\times[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_2-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{s-r-1}{k}\binom{r-1}{l}\int_0^{y_1}e^{-x_2}(1-e^{-x_2})^{\gamma-1}[1-(1-e^{-x_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1}\\&\times[1-[1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-s+k}\{\int_0^{x_2}e^{-x_1}(1-e^{-x_1})^{\gamma-1}[1-(1-e^{-x_1})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1}\\&\times[1-[1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{l+s-r-k-1}dx_1\}dx_2\end{aligned}$$

on solving which we get

$$I_2 = C_{r,s:n}\lambda^2\gamma^2 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^{\gamma}]^{\lambda-1}[1-(1-e^{-y_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_2-1}$$
$$\times[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_2-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{s-r-1}{k}\binom{r-1}{l}\left[\frac{(1-[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3})^{n-s+k+1}}{(n-s+k+1)}\right.$$
$$\left.-\frac{(1-[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3})^{n+l-r+1}}{(n+l-r+1)}\right]$$

If: $0 \le x_1 < x_2 < y_2 < y_1 \le \infty$; then

$$I_3 = \int_0^{y_2}\int_0^{x_2} f_1(y_1 \mid x_1) f_1(y_2 \mid x_2) f_{r,s:n}(x_1, x_2)\,dx_1 dx_2$$

Again using (1.9) and (2.5), we have

$$I_3 = C_{r,s:n}(\alpha_1+\alpha_3)^2\lambda^4\gamma^4 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^{\gamma}]^{\lambda-1}[1-(1-e^{-y_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_2-1}$$
$$\times[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_2-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{s-r-1}{k}\binom{r-1}{l}\int_0^{y_2} e^{-x_2}(1-e^{-x_2})^{\gamma-1}[1-(1-e^{-x_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1}$$
$$\times[1-[1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-s+k}\{\int_0^{x_2} e^{-x_1}(1-e^{-x_1})^{\gamma-1}[1-(1-e^{-x_1})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3-1}$$
$$\times[1-[1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{l+s-r-k-1}dx_1\}dx_2$$

on solving which we get

$$I_3 = C_{r,s:n}\lambda^2\gamma^2 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^{\gamma}]^{\lambda-1}[1-(1-e^{-y_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_2-1}$$
$$\times[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_2-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{s-r-1}{k}\binom{r-1}{l}$$
$$\times\left[\frac{(1-[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3})^{n-s+k+1}}{(n-s+k+1)}-\frac{(1-[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3})^{n+l-r+1}}{(n+l-r+1)}\right]$$

Case II*: When* $X > Y$

If: $0 \le y_1 < x_1 < y_2 < x_2 \le \infty$; then

$$I_4 = \int_{y_2}^{\infty}\int_{y_1}^{y_2} f_2(y_1 \mid x_1) f_2(y_2 \mid x_2) f_{r,s:n}(x_1, x_2)\,dx_1 dx_2$$

Using (1.10) and (2.5), we have

$$I_4 = C_{r,s:n}\alpha_1^{\,2}(\alpha_2+\alpha_3)^2\lambda^4\gamma^4 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^{\gamma}]^{\lambda-1}[1-(1-e^{-y_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{y_1})^{\gamma}\}^{\lambda}]^{\alpha_2+\alpha_3-1}$$
$$\times[1-\{1-(1-e^{y_2})^{\gamma}\}^{\lambda}]^{\alpha_2+\alpha_3-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{s-r-1}{k}\binom{r-1}{l}\int_{y_2}^{\infty} e^{-x_2}(1-e^{-x_2})^{\gamma-1}[1-(1-e^{-x_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda}]^{\alpha_1-1}$$
$$\times[1-[1-\{1-(1-e^{-x_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-s+k}\{\int_{y_1}^{y_2} e^{-x_1}(1-e^{-x_1})^{\gamma-1}[1-(1-e^{-x_1})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda}]^{\alpha_1-1}$$
$$\times[1-[1-\{1-(1-e^{-x_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{l+s-r-k-1}dx_1\}dx_2$$

on solving which we get

$$I_4 = C_{r,s:n}\alpha_1^2(\alpha_2+\alpha_3)^2\lambda^2\gamma^2 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^\gamma]^{\lambda-1}[1-(1-e^{-y_2})^\gamma]^{\lambda-1}[1-\{1-(1-e^{-y_1})^\gamma\}^\lambda]^{\alpha_2+\alpha_3-1}$$
$$\times[1-\{1-(1-e^{-y_2})^\gamma\}^\lambda]^{\alpha_2+\alpha_3-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}\sum_{g_1=0}^{l+s-r-k-1}\sum_{g_2=0}^{n-s+k}(-1)^{l+k+g_1+g_2}\binom{n-s+k}{g_2}\binom{l+s-r-k-1}{g_1}\binom{s-r-1}{k}\binom{r-1}{l}$$
$$\times\frac{1}{[\{(\alpha_1+\alpha_3)g_1+\alpha_1\}\{(\alpha_1+\alpha_3)g_2+\alpha_1\}]}[1-[1-\{1-(1-e^{-y_2})^\gamma\}^\lambda]^{(\alpha_1+\alpha_3)g_2+\alpha_1}][(1-\{1-(1-e^{-y_2})^\gamma\}^\lambda)^{(\alpha_1+\alpha_3)g_1+\alpha_1}$$
$$-(1-\{1-(1-e^{-y_1})^\gamma\}^\lambda)^{(\alpha_1+\alpha_3)g_1+\alpha_1}]$$

If: $0 \le y_1 < y_2 < x_1 < x_2 \le \infty$; then

$$I_5 = \int_{y_2}^{\infty}\int_{y_2}^{x_2} f_2(y_1 \mid x_1) f_2(y_2 \mid x_2) f_{r,s:n}(x_1,x_2)dx_1dx_2$$

Using (1.10) and (2.5), we have

$$I_5 = C_{r,s:n}\alpha_1^2(\alpha_2+\alpha_3)^2\lambda^4\gamma^4 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^\gamma]^{\lambda-1}[1-(1-e^{-y_2})^\gamma]^{\lambda-1}[1-\{1-(1-e^{-y_1})^\gamma\}^\lambda]^{\alpha_2+\alpha_3-1}$$
$$\times[1-\{1-(1-e^{-y_2})^\gamma\}^\lambda]^{\alpha_2+\alpha_3-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{s-r-1}{k}\binom{r-1}{l}\int_{y_2}^{\infty}e^{-x_2}(1-e^{-x_2})^{\gamma-1}[1-(1-e^{-x_2})^\gamma]^{\lambda-1}[1-\{1-(1-e^{-x_2})^\gamma\}^\lambda]^{\alpha_1-1}$$
$$\times[1-[1-\{1-(1-e^{-x_2})^\gamma\}^\lambda]^{\alpha_1+\alpha_3}]^{n-s+k}\{\int_{y_2}^{x_2}e^{-x_1}(1-e^{-x_1})^{\gamma-1}[1-(1-e^{-x_1})^\gamma]^{\lambda-1}[1-\{1-(1-e^{-x_1})^\gamma\}^\lambda]^{\alpha_1-1}$$
$$\times[1-[1-\{1-(1-e^{-x_1})^\gamma\}^\lambda]^{\alpha_1+\alpha_3}]^{l+s-r-k-1}dx_1\}dx_2$$

After some computation we get

$$I_5 = C_{r,s:n}\alpha_1^2(\alpha_2+\alpha_3)^2\lambda^2\gamma^2 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^\gamma]^{\lambda-1}[1-(1-e^{-y_2})^\gamma]^{\lambda-1}[1-\{1-(1-e^{-y_1})^\gamma\}^\lambda]^{\alpha_2+\alpha_3-1}$$
$$\times[1-\{1-(1-e^{-y_2})^\gamma\}^\lambda]^{\alpha_2+\alpha_3-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}\sum_{g_1=0}^{l+s-r-k-1}\sum_{g_2=0}^{n-s+k}\frac{(-1)^{l+k+g_1+g_2}}{[(\alpha_1+\alpha_3)g_1+\alpha_1]}\binom{n-s+k}{g_2}\binom{l+s-r-k-1}{g_1}\binom{s-r-1}{k}\binom{r-1}{l}$$
$$\times\left[\left\{\frac{1-[1-\{1-(1-e^{-y_2})^\gamma\}^\lambda]^{(\alpha_1+\alpha_3)g_1+(n-s+k)g_2+2\alpha_1}}{(\alpha_1+\alpha_3)g_1+(n-s+k)g_2+2\alpha_1}\right\}-[1-\{1-(1-e^{-y_2})^\gamma\}^\lambda]^{(\alpha_1+\alpha_3)g_1+\alpha_1}\left\{\frac{1-[1-\{1-(1-e^{-y_2})^\gamma\}^\lambda]^{(n-s+k)g_2+\alpha_1}}{(n-s+k)g_2+\alpha_1}\right\}\right]$$

If: $0 \le y_2 < y_1 < x_1 < x_2 \le \infty$; then

$$I_6 = \int_{y_1}^{\infty}\int_{y_1}^{x_2} f_2(y_1 \mid x_1) f_2(y_2 \mid x_2) f_{r,s:n}(x_1,x_2)dx_1dx_2$$

Again on using (1.10) and (2.5), we have

$$I_6 = C_{r,s:n}\alpha_1^2(\alpha_2+\alpha_3)^2\lambda^4\gamma^4 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^\gamma]^{\lambda-1}[1-(1-e^{-y_2})^\gamma]^{\lambda-1}[1-\{1-(1-e^{-y_1})^\gamma\}^\lambda]^{\alpha_2+\alpha_3-1}$$
$$\times[1-\{1-(1-e^{-y_2})^\gamma\}^\lambda]^{\alpha_2+\alpha_3-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{s-r-1}{k}\binom{r-1}{l}\int_{y_1}^{\infty}e^{-x_2}(1-e^{-x_2})^{\gamma-1}[1-(1-e^{-x_2})^\gamma]^{\lambda-1}[1-\{1-(1-e^{-x_2})^\gamma\}^\lambda]^{\alpha_1-1}$$
$$\times[1-[1-\{1-(1-e^{-x_2})^\gamma\}^\lambda]^{\alpha_1+\alpha_3}]^{n-s+k}\{\int_{y_1}^{x_2}e^{-x_1}(1-e^{-x_1})^{\gamma-1}[1-(1-e^{-x_1})^\gamma]^{\lambda-1}[1-\{1-(1-e^{-x_1})^\gamma\}^\lambda]^{\alpha_1-1}$$
$$\times[1-[1-\{1-(1-e^{-x_1})^\gamma\}^\lambda]^{\alpha_1+\alpha_3}]^{l+s-r-k-1}dx_1\}dx_2$$

on solving which we get

$$I_6 = C_{r,s:n}\alpha_1^{\,2}(\alpha_2+\alpha_3)^2\lambda^2\gamma^2 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^{\gamma}]^{\lambda-1}[1-(1-e^{-y_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_2+\alpha_3-1}$$
$$\times[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_2+\alpha_3-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}\sum_{g_1=0}^{l+s-r-k-1}\sum_{g_2=0}^{n-s+k}\frac{(-1)^{l+k+g_1+g_2}}{[(\alpha_1+\alpha_3)g_1+\alpha_1]}\binom{n-s+k}{g_2}\binom{l+s-r-k-1}{g_1}\binom{s-r-1}{k}\binom{r-1}{l}$$
$$\times\left[\left\{\frac{1-[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{(\alpha_1+\alpha_3)(g_1+g_2)+2\alpha_1}}{(\alpha_1+\alpha_3)(g_1+g_2)+2\alpha_1}\right\}-\left\{\frac{[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{(\alpha_1+\alpha_3)g_1+\alpha_1}-[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{(\alpha_1+\alpha_3)(g_1+g_2)+2\alpha_1}}{(\alpha_1+\alpha_3)g_2+\alpha_1}\right\}\right]$$

Case III: When $X = Y$; then

$$I_7 = f_3(y_1 \mid x_1) f_3(y_2 \mid x_2) f_{r,s:n}(x_1, x_2)$$

On using (1.11) and (2.5), we have

$$I_7 = C_{r,s:n}\alpha_3^{\,2}\lambda^2\gamma^2 e^{-(y_1+y_2)}(1-e^{-y_1})^{\gamma-1}(1-e^{-y_2})^{\gamma-1}[1-(1-e^{-y_1})^{\gamma}]^{\lambda-1}[1-(1-e^{-y_2})^{\gamma}]^{\lambda-1}[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_2+\alpha_3-1}$$
$$\times[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_2+\alpha_3-1}\sum_{l=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{l+k}\binom{s-r-1}{k}\binom{r-1}{l}[1-[1-\{1-(1-e^{-y_1})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{l+s-r-k-1}$$
$$\times[1-[1-\{1-(1-e^{-y_2})^{\gamma}\}^{\lambda}]^{\alpha_1+\alpha_3}]^{n-s+k}$$

After adding $I_1,\ldots,I_7$ we get the required joint density of two concomitant of order statistics.

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