International Journal of Computational and Theoretical Statistics
ISSN (2210-1519)
Int. J. Comp. Theo. Stat. 5, No. 2 (Nov-2018)

http://dx.doi.org/10.12785/ijcts/050202

A New Extension of the Exponential Power Distribution with Application to Lifetime Data

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Received February 10, 2018, Revised June 5, 2018, Accepted June 27, 2018 Published November 1, 2018

Abstract: This paper derives a new four-parameter generalized exponential power lifetime probability model for life time data, which generalizes some well-known exponential power lifetime distributions. It is observed that our proposed new distribution bears most of the properties of skewed distributions in reliability and life testing context. It is skewed to the right as well as its failure rate function has the increasing and bathtub shape behaviors. The estimation of the parameters, and simulation and applications of the proposed model have also been discussed.

Keywords: Exponential power distribution, Estimation, F^{α} distribution, Generalized exponential power distribution, Lifetime, Reliability, Simulation.

1. INTRODUCTION

In recent years, it has been found that increasing the number of parameters make the well-known distributions more flexible so that the enhanced models would be more applicable and suitable for certain lifetime data when usual models are not. For example, Barriga et al. [6] proposed a new three-parameter extension of Smith and Bain [32] exponential power lifetime distribution, by introducing an additional shape parameter, known in the literature as the complementary exponential power (CEP) distribution, which mainly generalizes the exponential power distribution of Smith and Bain [34]. As defined by Smith and Bain [34], a positive continuous random variable X is said to have an exponential power distribution with scale parameter $\lambda > 0$ and shape parameter $\kappa > 0$, which we denote by X ~ *exponential-power* (λ, κ), if its pdf is given by

$$f(x) = \kappa \lambda^{\kappa} x^{\kappa - 1} e^{\lambda^{\kappa} x^{\kappa}} e^{1 - e^{\lambda^{\kappa} x^{\kappa}}}, \quad x > 0 . \tag{1.1}$$

Since the exponential power distribution (1.1) was introduced by Smith and Bain [34], many authors and researchers have studied various properties of this distribution, among them, Leemis [17], Rajarshi and Rajarshi [30], and Chen [7, 8], are notable.

Further, it should be noted that, in recent years, the F^{α} or exponentiated power distributions have been widely studied because of their wide applicability in the modeling and analysis of life time data. Many researchers and authors have developed various classes of F^{α} distributions; see, for example, Ahuja and Nash [3], Mudholkar and Srivastava [21], Mudholkar et al. [22], Mudholkar and Hutson [20], Gupta and Kundu [12,13], Nassar and Eissa [26], Nadarajah and Gupta [23], Pal et al. [28], Nadarajah and Kotz [24], Cho et al. [9], Shawky and Abu-Zinadah [33], Persson and Ryden [29], Shadrokh and Pazira [31], Nadarajah [25] , and Lemonte and Cordeiro [18], Shakil and Ahsanullah [32], Hamedani [14], and Al-Hussaini and Ahsanullah [4], among others.

A positive continuous random variable X is said to have F^{α} distribution if its cdf is given by $G(x) = [F(x)]^{\alpha}$, $\alpha > 0$, $x \geq 0$, which is the αth power of the base line distribution function $F(x)$. The distribution $G(x)$ is also

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called an exponentiated distribution of the given base cdf $F(x)$. Its probability density function (pdf) $g(x)$ is given by $g(x)=\alpha f(x)F^{\alpha-1}(x)$. The F^{α} distribution is also called the proportional reverse hazard rate model (PRHRM). The reverse hazard rate function (rhrf) of $G(x)$ is given by $\lambda_G^*(x)=\frac{d}{dx}(\ln(G(x)))=\frac{g(x)}{G(x)}$, where $g(x)$ is the pdf corresponding to $G(x)$. Thus

$$\lambda_G^*(x)=\frac{\alpha(F(x))^{\alpha-1}f(x)}{(F(x))^{\alpha}}=\alpha\,\lambda_F^*(x). \tag{1.2}$$

F^{α} is also called the Lehmann alternatives, that is, the model of non-parametric class of alternatives, see Lehmann [15].

In recent years, many authors have studied classical distributions, such as the exponential and the exponentiated power Lindley (EPL) distributions, among others,by adding new parameters to the baseline distributions, in order to obtain more flexibility, and for building meaningful distributions. See, for example, Alizadeh et al. [4] and Lemonte et al. [19], and references therein. Thus, in the present paper, motivated by the importance of the F^{α} distributions, and the fact that increasing the number of parameters make the well-known distributions more flexible and useful, we derive a four-parameter probability model for lifetime data, which generalizes the exponential power distribution of Smith and Bain [34], two-parameter lifetime distribution of Chen [8], and the complementary exponential power distribution of Barriga et al. [6]. We call it as the F^{α} generalized exponential power lifetime probability model, or F^{α} GEP Model.

The organization of the paper is as follows. In Section 2, we introduce the proposed model, and discuss its distributional and reliability properties. In Section 3, we provide estimation of parameters. The simulation and applications of the proposed model have been discussed in Sections 4 and 5 respectively. We provide some concluding remarks in Section 6.

2. THE PROPOSED NEW LIFETIME EXPONENTIAL POWER DISTRIBUTION

In this section, using the F^{α} scheme, we propose a new four-parameter probability model for life time data (F^{α} GEP Model), and discuss its distributional and reliability properties, as described below.

2.1. The Probability Density and Cumulative Distribution Functions:

For a positive continuous random variable X with scale parameter $\lambda,\theta>0$ and shape parameters $k,\alpha>0$, we define its pdf by

$$f(x)=\alpha\theta k\lambda^k x^{k-1}e^{\lambda^k x^k}e^{\theta\left(1-e^{\lambda^k x^k}\right)}\left[1-e^{\theta\left(1-e^{\lambda^\kappa x^\kappa}\right)}\right]^{\alpha-1}. \tag{2.1}$$

The cumulative distribution function (cdf) corresponding to pdf (2.1) is given by

$$F(x)=\left[1-e^{\theta\left(1-e^{\lambda^\kappa x^\kappa}\right)}\right]^{\alpha},\quad x>0\ . \tag{2.2}$$

It is easily seen that, by proper choice of parameters, the pdf (2.1) contains the exponential power distribution of Smith and Bain [34], Chen [8], and Barriga et al. [6], among some other well-known distributions.

2.2. Distributional Properties

2.2.1. Reliability:

It is easy to see from the pdf in (2.1) that our proposed distribution is an increasing failure rate (IFR) distribution. For different combinations of values of the parameters, the graphs of the pdf (2.1) are shown in Figure 1 below. In view of these graphs, the proposed distribution appears to be unimodal and right skewed.

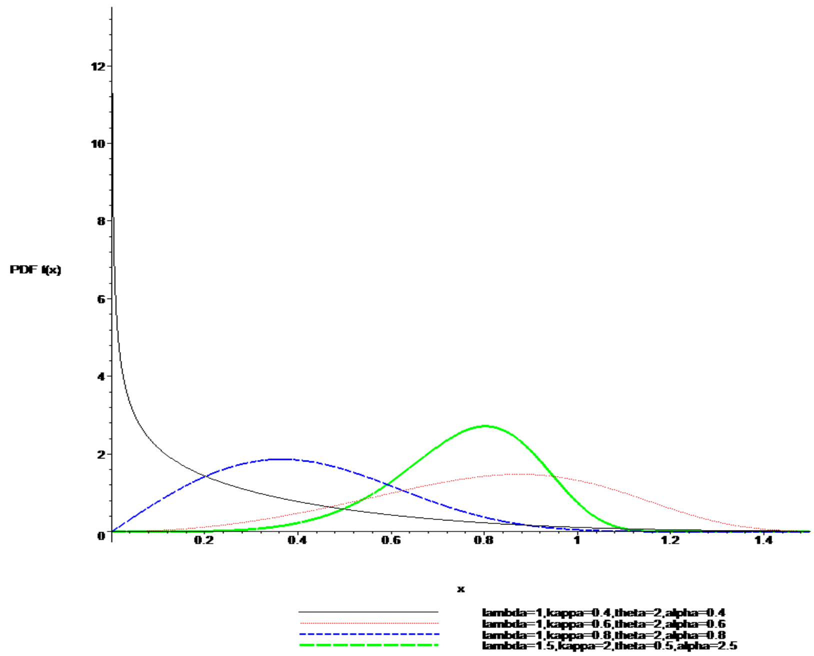

Figure 1. For pdf (2.1)

Recalling the definition of the hazard (or failure) rate for non-repairable populations as the instantaneous rate of failure for the survivors to time, say, x, during the next instant of time, the survival (or reliability) and the hazard (or failure) rate functions of the proposed distribution are respectively given by

$$R(x) = 1 - F(x) = 1 - \left[1 - e^{\theta\left(1 - e^{\lambda^{\kappa} x^{\kappa}}\right)}\right]^{\alpha}, \tag{2.3}$$

and

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\alpha\,\theta\,k\,\lambda^{k} x^{k-1} e^{\lambda^{k} x^{k}} e^{\theta\left(1 - e^{\lambda^{k} x^{k}}\right)} \left[1 - e^{\theta\left(1 - e^{\lambda^{\kappa} x^{\kappa}}\right)}\right]^{\alpha - 1}}{1 - \left[1 - e^{\theta\left(1 - e^{\lambda^{\kappa} x^{\kappa}}\right)}\right]^{\alpha}}. \tag{2.4}$$

Further, it is also sometimes useful to find the average failure rate function (AFR), over any interval, say, $(0, t)$, that averages the failure rate over the interval, $(0, t)$, see, for example, Barlow and Proschan [5]. Thus, for the proposed distribution, it is given by

$$AFR(0, t) = AFR(t) = -\frac{1}{t}\ln(R(t)) = -\frac{1}{t}\ln\left(1 - \left[1 - e^{\theta\left(1 - e^{\lambda^{\kappa} t^{\kappa}}\right)}\right]^{\alpha}\right),$$

which in view of the expansion of logarithmic function as a power series, is seen to be positive irrespective of the values of scale parameters and hence the proposed distribution is Increasing Failure Rate on Average (IFRA). Also recall that a life distribution $F(.)$ is NBU (new better than used) if $R(x+y) \le R(x)R(y),\ \forall\ x, y \ge 0$, and NWU (new worse than used) if the reversed inequality holds, see, for example, Barlow and Proschan [4]. We note that, for the proposed distribution, since

$$R(x+y) = 1 - \left[1 - e^{\theta\left(1 - e^{\lambda^{\kappa}(x+y)^{\kappa}}\right)} \right]^{\alpha}$$

and

$$R(x).R(y) = \left(1 - \left[1 - e^{\theta\left(1 - e^{\lambda^{\kappa} x^{\kappa}}\right)} \right]^{\alpha} \right) . \left(1 - \left[1 - e^{\theta\left(1 - e^{\lambda^{\kappa} y^{\kappa}}\right)} \right]^{\alpha} \right) ,$$

it is easy to see that $R(x+y) \le R(x).R(y)$, which implies that the proposed distribution has the property of Never Better than Used (NBU). Differentiating Eq. (2.4) with respect to x, we have

$$h'(x) = \tfrac{f'(x)}{f(x)} h(x) + [h(x)]^2 = \Big[\, k\lambda^k x^{k-1} + \frac{k-1}{x} - \theta k \lambda^k x^{k-1} e^{\lambda^k x^k} + \frac{(\alpha-1)\theta k \lambda^k x^{k-1} e^{\lambda^k x^k} e^{\theta\left(1-e^{\lambda^k x^k}\right)}}{1 - e^{\theta\left(1-e^{\lambda^{\kappa} x^{\kappa}}\right)}} \, \Big] h(x) + [h(x)]^2 , \tag{2.5}$$

for $x > 0$. In order to discuss the behavior of the failure rate function, $h(x)$, let $h'(x) = 0$. We observe that the nonlinear equation $h'(x) = 0$ does not have a closed form solution, but could be solved numerically by using some mathematical software such as Maple and R. It is obvious from Eq. (2.5) that $h'(x)$ is positive irrespective of the values of the parameters. This further confirms the IFR property of our proposed model and $k, \alpha > 0$ being the shape parameters, the failure rate function, $h(x)$, may have a bathtub shape when $k > 1$ or $\alpha > 1$, and may be increasing when $0 < k \le 1$ or $0 < \alpha \le 1$. For some special values of the parameters, the graphs of the hazard function (2.4) are illustrated in Figures $2-3$ below. The effects of the parameters are obvious from these figures. The increasing and bathtub shape behaviors of the failure rate function, $h(x)$, are also evident from these Figures.

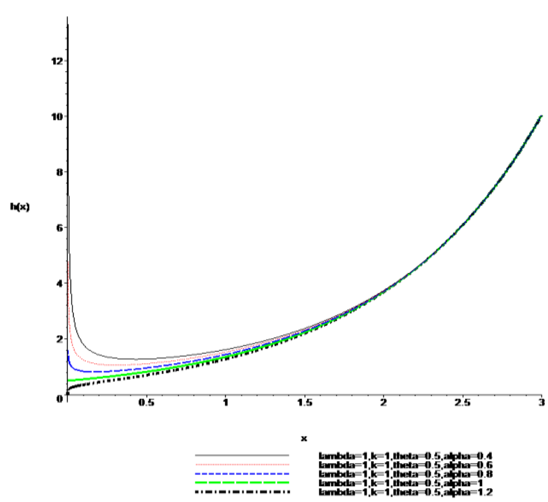

Figure 2. HAZARD RATE FUNCTION, $h(x)$, of Eq. (2.4), when $\lambda = 1$, $\theta = 0.5$, $k = 1$, and $\alpha = 0.4, 0.6, 0.8, 1, 1.2$.

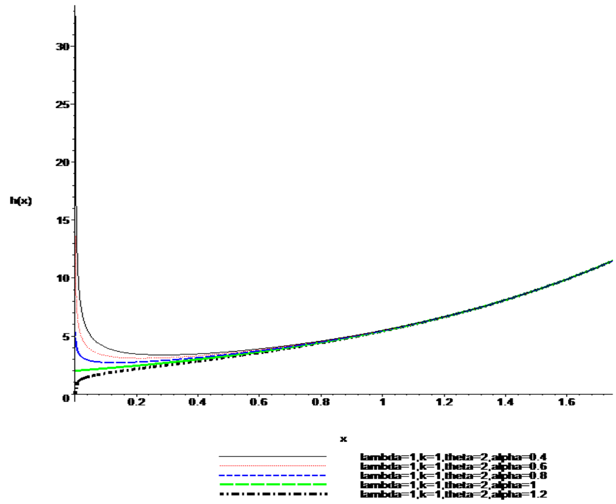

Figure 3. HAZARD RATE FUNCTION, $h(x)$, of Eq. (2.4), when $\lambda=1, \theta=2, k=1$, and $\alpha=0.4, 0.6, 0.8, 1, 1.2$.

2.2.2. Moments:

In what follows, for the random variable X having the pdf $f(x)$ as in Eq. (2.1), we derive the nth moment, $E(X^n)$, where $n>0$ is an integer. We have

$$E(X^n)=\int_0^\infty x^n\, f_X(x)dx$$

$$=\int_0^\infty x^n\left[\alpha\,\theta\,k\,\lambda^k x^{k-1} e^{\lambda^k x^k} e^{\theta\left(1-e^{\lambda^k x^k}\right)}\left[1-e^{\theta\left(1-e^{\lambda^\kappa x^\kappa}\right)}\right]^{\alpha-1}\right]dx. \quad (2.6)$$

Letting $1-e^{\theta\left(1-e^{\lambda^\kappa x^\kappa}\right)}=u$ in Eq. (2.5), and simplifying, we have

$$E(X^n)=\frac{\alpha}{\lambda^n}\int_0^1 u^{\alpha-1}\left[\ln(\theta-\ln(1-u))-\ln\theta\right]^{\frac{n}{k}} du. \quad (2.7)$$

Now, taking $1-u=t$ in Eq. (2.6), and then using the binomial series representation $(1+w)^s=\sum_{j=0}^{\infty}\frac{(-1)^j (s)_j\, w^j}{j!}$, for any real $s\neq 0$, where $(s)_j=\frac{\Gamma(s+j)}{\Gamma(s)}=s(s+1)\ldots(s+j-1)$ denotes the Pochhammer symbol, we easily obtain the nth moment, $E(X^n)$, given by

$$E(X^n)=\frac{\alpha}{\lambda^n}\sum_{j=0}^{\infty}\frac{(\alpha-1)_j}{j!}\int_0^1 t^j\left[\ln(\theta-\ln t)-\ln\theta\right]^{\frac{n}{k}} dt. \quad (2.8)$$

It is obvious from Eq. (2.7) that, when $n=1$, the $1st$ moment, $E(X)$, of X, is mathematically easily tractable for $k=1$. Hence, by taking $n=1$ and $k=1$ in Eq. (2.7), and using Gradshteyn and Ryzhik [11], Eq. 4.326.1, Page 572, for the integral in Eq. (2.7) , that is,

$\int_0^1 [\ln(a - \ln z)] z^{\mu-1} \, dz = \frac{1}{\mu}\left[\ln a - e^{a\mu} \, Ei(-a\mu)\right], \, \Re e\,(\mu) > 0, \, \Re e\,(a) > 0$, and simplifying, we easily obtain the $1st$ moment, $E\,(X)$, given by

$$E\,(X) = \frac{\alpha}{\lambda} \sum_{j=0}^{\infty} \frac{(\alpha-1)_j}{(j+1)!} \left[- e^{\theta\,(j+1)} \, Ei(-\theta(j+1))\right], \; \alpha, \lambda, \theta > 0, \qquad (2.9)$$

where $Ei(z)$, known as the exponential-integral function, and, for $z > 0$, is defined as follows:

$$Ei(z) = \gamma + \ln z + \sum_{k=1}^{\infty} \frac{z^k}{k.k!}, \quad z > 0,$$

where $\gamma = -\psi(1) \approx 0.577216$ denotes the Euler's constant ; (see, for example, Gradshteyn and Ryzhik [11], Abramowitz and Stegun [2], and Oldham et al. [27], among others).

3. Estimation of Parameters

3.1. The Method of Moments: From the nth moment (2.6), $E\,(X^n)$, of the proposed model, taking $n = 1, 2, 3, 4$, and evaluating the respective integrals numerically, we obtain the first four moments. Then, in view of the moment equation (2.6) depends on the exponential-integral function, $Ei(-z)$, the moment estimation (MMEs) of the parameters α , λ , θ and k can be determined by solving the system of four equations obtained from (2.6) by Newton-Raphson's iteration method, and using the computer package such as Maple, or R, MathCAD, or other software.

3.2. The Method of Maximum Likelihood: Given a sample $\{x_i\}$, $i = 1, 2, 3, \ldots, n$, the likelihood function of (2.1) is given by $L = \prod_{i=1}^{n} f\,(x_i)$. The objective of the likelihood function approach is to determine those values of the parameters that maximize the function L . Suppose $R = \ln(L) = \sum_{i=1}^{n} \ln[f\,(x_i)]$. Then, upon differentiation, the following system of equations is obtained:

$$\frac{\partial R}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left(1 - e^{\theta\left(1 - e^{\left(\lambda^{\kappa} x_i^{\kappa}\right)}\right)}\right) = 0, \qquad (3.1)$$

$$\frac{\partial R}{\partial \lambda} = \frac{nk}{\lambda} + \frac{\lambda^k \left(\sum_{i=1}^{n} x_i^{\,k}\right)}{\lambda} - \frac{\theta k \lambda^k \left(\sum_{i=1}^{n} x_i^{\,k}\right) e^{\left(\lambda^k\left(\sum_{i=1}^{n} x_i^{\,k}\right)\right)}}{\lambda}$$

$$+(\alpha-1)\left[\sum_{i=1}^{n}\frac{\theta k\lambda^{k}x_i^{\,k}e^{\left(\lambda^{\kappa}x_i^{\kappa}\right)}e^{\theta\left(1-e^{\lambda^{\kappa}x_i^{\kappa}}\right)}}{\lambda\left(1-e^{\theta\left(1-e^{\left(\lambda^{\kappa}x_i^{\kappa}\right)}\right)}\right)}\right]=0, \quad (3.2)$$

$$\frac{\partial R}{\partial\theta}=\frac{n}{\theta}+n-e^{\left(\lambda^{k}\left(\sum_{i=1}^{n}x_i^{k}\right)\right)}+(\alpha-1)\left[\sum_{i=1}^{n}\left(-\frac{\left(1-e^{\left(\lambda^{\kappa}x_i^{\kappa}\right)}\right)e^{\theta\left(1-e^{\lambda^{\kappa}x_i^{\kappa}}\right)}}{1-e^{\theta\left(1-e^{\left(\lambda^{\kappa}x_i^{\kappa}\right)}\right)}}\right)\right]=0, \quad (3.3)$$

$$\frac{\partial R}{\partial k}=n\ln(\lambda)+\left(\sum_{i=1}^{n}\ln(x_i)\right)+\lambda^{k}\ln(\lambda)\left(\sum_{i=1}^{n}x_i^{\,k}\right)+\lambda^{k}\left(\sum_{i=1}^{n}x_i^{\,k}\ln(x_i)\right)$$

$$-\theta\left[\lambda^{k}\ln(\lambda)\left(\sum_{i=1}^{n}x_i^{\,k}\right)+\lambda^{k}\left(\sum_{i=1}^{n}x_i^{\,k}\ln(x_i)\right)\right]e^{\left(\lambda^{k}\left(\sum_{i=1}^{n}x_i^{k}\right)\right)}$$

$$+(\alpha-1)\left[\sum_{i=1}^{n}\frac{\theta\left(\lambda^{k}\ln(\lambda)\right)x_i^{\,k}+\lambda^{k}x_i^{\,k}\ln(x_i)e^{\left(\lambda^{\kappa}x_i^{\kappa}\right)}e^{\theta\left(1-e^{\lambda^{\kappa}x_i^{\kappa}}\right)}}{\left(1-e^{\theta\left(1-e^{\left(\lambda^{\kappa}x_i^{\kappa}\right)}\right)}\right)}\right]=0. \quad (3.4)$$

Then the maximum likelihood estimates (MLE) of the parameters α , λ , θ and k are obtained by solving the maximum likelihood equations (3.1 - 3.4), that is, by solving the equation

$\frac{\partial R}{\partial\alpha}=0$, $\frac{\partial R}{\partial\beta}=0$, $\frac{\partial R}{\partial\nu}=0$, and $\frac{\partial R}{\partial p}$, applying the Newton-Raphson's iteration method and using the computer package such as Maple, or R, MathCAD14, or other software.

4. SIMULATION:

In this section, we use simulation to compare the performances of the different methods of estimation mainly with respect to their biases and mean square errors (MLEs) for different sample sizes. A numerical study is performed using MathCAD14 software. Different sample sizes are considered through the experiments at size n = 15, 20, 25, 30, 50 and 100 for different values of parameters, viz., $\alpha=2$, $\theta=1.573$, $\lambda=0.179$, and $k=0.4$, which we have chosen arbitrarily. The experiment is repeated 1000 times. In each experiment, the estimates of the parameters are obtained by two methods of estimation: MME and MLE respectively. The means, MSEs and biases for the different estimators are reported from these experiments in Tables 1 and 2 respectively below.

Table 1. The parameter estimation from New Model using MME
($\alpha = 2$, $\theta = 1.573$, $\lambda = 0.179$, and $k = 0.4$)

n		Mean	Bias	MSE	SE	n		Mean	Bias	MSE	SE
15	α	2.30189	0.30189	1.59374	1.226	30	α	2.05141	0.05141	0.20795	0.453
	θ	1.58888	0.01588	0.00905	0.094		θ	1.57545	0.00245	0.00397	0.063
	k	0.42625	0.02625	0.00757	0.083		k	0.40654	0.00654	0.00124	0.035
	λ	0.22245	0.04345	0.00371	0.043		λ	0.21055	0.03155	0.00234	0.037
20	α	2.21005	0.21005	0.91293	0.932	50	α	2.03987	0.03987	0.07912	0.278
	θ	1.58043	0.00743	0.00614	0.078		θ	1.56967	-0.00333	0.00172	0.041
	k	0.41539	0.01539	0.00374	0.059		k	0.40025	0.00025	0.00033	0.018
	λ	0.21736	0.03836	0.00355	0.046		λ	0.20232	0.02332	0.00144	0.03
25	α	2.1493	0.1493	0.79453	0.879	100	α	2.03937	0.03937	0.04285	0.203
	θ	1.57836	0.00536	0.00509	0.071		θ	1.56751	-0.00549	0.00091	0.03
	k	0.40856	0.00856	0.00195	0.043		k	0.39855	-0.00145	0.00010	0.0097
	λ	0.21184	0.03284	0.00250	0.038		λ	0.19382	0.01482	0.00083	0.025

Table 2. The parameter estimation from New Model using MLE
($\alpha = 2$, $\theta = 1.573$, $\lambda = 0.179$, and $k = 0.4$)

n		Mean	Bias	MSE	SE	n		Mean	Bias	MSE	SE
15	α	3.1471	1.1471	19.62337	4.279	30	α	2.78639	0.78639	12.31686	3.42
	θ	2.42229	0.84929	7.15832	2.537		θ	2.13412	0.56112	3.18027	1.693
	k	0.30556	-0.09444	1.82558	1.348		k	0.31979	-0.08021	0.07893	0.269
	λ	0.17439	-0.00461	0.79922	0.894		λ	0.16906	-0.00994	0.03920	0.198
20	α	2.85685	0.85685	14.01596	3.644	50	α	2.50295	0.50295	6.60940	2.521
	θ	2.28438	0.71138	6.15260	2.376		θ	1.82749	0.25449	2.32187	1.502
	k	0.31515	-0.08485	0.12457	0.343		k	0.3095	-0.0905	0.07165	0.252

	λ	0.17736	-0.00164	0.09722	0.312		λ	0.14871	-0.03029	0.04341	0.206
25	α	2.94226	0.94226	26.11667	5.023	**100**	α	2.4348	0.4348	13.31539	3.623
	θ	2.20758	0.63458	5.06740	2.16		θ	1.62606	0.05306	0.54144	0.734
	k	0.31637	-0.08363	0.11511	0.329		k	0.33984	-0.06016	0.04774	0.21
	λ	0.16777	-0.01123	0.03753	0.193		λ	0.15737	-0.02163	0.01093	0.102

If we review Tables 1 and 2, it is observed that as the sample size increases, both absolute bias and MSE decrease and converge close to true values of parameters.

5. APPLICATIONS:

An Example on Average Annual Percent Change in Private Health Insurance Premiums: In this section, we use a real data set to illustrate the potentiality and the performance of the new model by considering the Average Annual Percent Change in Private Health Insurance Premiums (All Benefits: Health Services and Supplies), Calendar Years 1969-2007 (SOURCE: Centers for Medicare & Medicaid Services, Office of the Actuary, National Health Statistics Group), as provided in the following Table 3. We compare the new distribution with other distributions, namely, transmuted quasi Lindley distribution (TQL) (see Elbatal and Elgarhy [10]), beta Weibull (BW) (see Lee et al., [16]), Burr XII, and exponential power (k,λ) of Smith and Bain [34].

Table 3. Average Annual Percent Change in Private Health Insurance Premiums

14.4, 14.0, 15.4, 9.4, 11.7, 15.0, 24.9, 20.7, 12.5, 14.9, 12.6, 16.7, 13.8, 11.0, 12.9, 10.1, 1.9, 8.5, 16.5, 15.3, 13.3, 9.8, 8.4, 7.9, 3.7, 5.1, 4.6, 4.4, 5.4, 6.1, 8.0, 10.0, 11.2, 10.1, 6.4, 6.7, 5.7, 5.8

The mean, median, skewness and excess kurtosis of this data are 10.7, 10.1, 0.603 and 0.557, respectively. We can see that the data is right skewed. The estimate of the unknown parameters of each distribution is obtained by the maximum-likelihood method. In order to compare the new distribution model with the three distribution models, as stated above, various criteria were used. Criteria like $-2\ln L$, Akaike information criterion (AIC), Bayesian information criterion (BIC), the correct Akaike information criterion ($CAIC$), Hannan information criterion ($HQIC$), the Kolmogorov-Smirnov ($K-S$) and $p-value$ statistics are considered for the data set. The formulas for these criteria are as follows:

$$AIC = 2k - 2\ln L, \qquad CAIC = AIC + \frac{2k(k+1)}{n-k-1},$$

$$BIC = k\ln(n) - 2\ln L, \qquad HQIC = 2k\ln\left[\ln(n)\right] - 2\ln L,$$

$$k-s = \sup_y\left[F_n(\mathrm{y}) - F(\mathrm{y})\right],$$

where k is the number of parameters in the statistical model, n is the sample size and $\ln L$ is the maximized value of the log- likelihood function under the considered model, $F_n(\mathrm{y})$ is the empirical distribution function, and $F(\mathrm{y})$ denotes the cdf for each distribution. The "best" distribution corresponds to the smallest values of $-2\ln L$, AIC, BIC, $CAIC$, $HQIC$, $K-S$, and the biggest value of $p-value$ criteria. Using MathCAD 14 software, the estimation of the parameters and goodness-of-fit are provided below in Tables 4 and 5 below. Table 4 shows the MLEs of the model parameters and its standard error (S.E) (in parentheses) for data set. Table 5 gives the values values of mesurments for our considered data set.

Table 4. The MLEs and S.E of the model parameters for the data set (Table 3)

Distribution	MLEs and S. E			
New Model $(\alpha,\theta,k,\lambda)$	3.604 (0.06649)	1.394 (0.037)	0.834 (0.223)	0.077 (0.038)
TQL (β,γ,δ)	0.066 (0.00625)	86.092 (0.15)	0.468 (0.139)	-
BW (a,b,η,μ)	32.934 (0.429)	0.145 (0.129)	0.994 (0.18)	1.006 (0.033)
Burr XII (c,ρ)	0.038 (0.01071)	11.804 (0.059)	-	-
Exponential power (k,λ)			1.578 (0.01071)	0.063 (0.059)

Table 5. Measurements for all models based on for the data set (Table 3)

Distribution	$-2\ln L$	AIC	BIC	$CAIC$	$HQIC$	$K-S$	$p-value$
New Model $(\alpha,\theta,k,\lambda)$	293.222	301.222	299.541	302.434	303.553	0.06988	0.99246
TQL (β,γ,δ)	373.508	379.508	378.248	380.214	381.256	0.28566	0.00405
BW (a,b,η,μ)	357.283	365.283	363.602	366.495	367.613	0.15618	0.3121
Burr XII (c,ρ)	481.319	485.319	484.478	485.662	486.484	0.43034	0.000002
Exponential power (k,λ)	326.226	330.226	329.386	330.569	331.392	0.10161	0.82758

If we review the computed values in Tables 4 and 5, it is observed that our proposed distribution is a strong competitor to other distributions used here for fitting to the considered data set (Table 3). A density plot is used to compare the fitted densities of the models with the empirical histogram of the observed data. Figures 4 – 5 below provide the plots of estimated cumulative and estimated densities of the fitted New Model, TQL, BW and Burr XII models for the data set (Table 3).

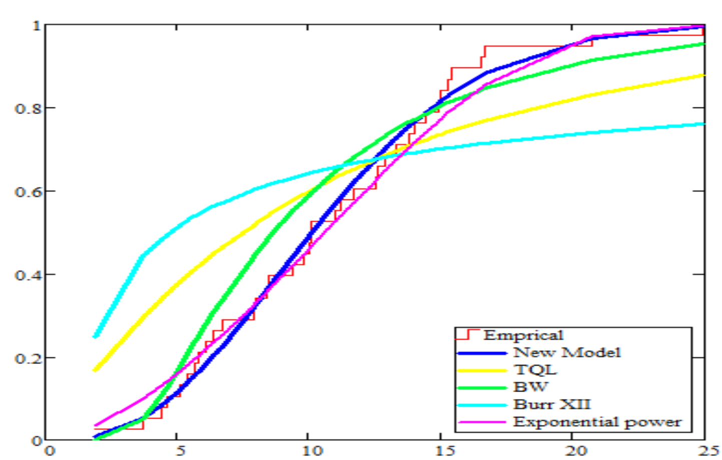

Figure 4. Estimated cumulative densities of the models for data set (Table 3).

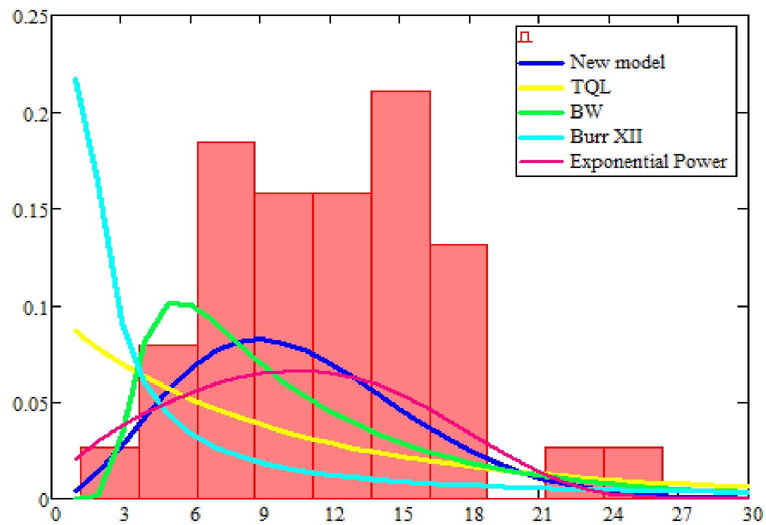

Figure 5. Estimated densities of the models for data set (Table 3).

It is observed from Figures 4 – 5 that the fitted density for the the new model is the closest to the empirical histogram in comparison to the other fitted models.

6. CONCLUDING REMARKS:

In this paper, we have derived a new four-parameter probability model for life time data, using an F^{α} scheme, which generalizes the Smith-Bain's exponential power distribution [32] as well as of Chen's bathtub shape or increasing failure rate model [8]. It is observed that our proposed new distribution is skewed to the right and bears most of the properties of skewed distributions, and is more flexible and is a natural generalization of many well-known life distributions. To illustrate the MOM and MLE techniques, we have arbitrary selected the following values of the parameters $\alpha = 2$, $\theta = 1.573$, $\lambda = 0.179$, and $k = 0.4$. Based on these, it is observed that our proposed distribution is a strong competitor to other distributions, namely, transmuted quasi Lindley distribution (TQL) of Elbatal and Elgarhy [10], beta Weibull (BW) of Lee et al. [16], Burr XII, and exponential power (k, λ) of Smith and Bain [34], used here for fitting to the considered data set (Table 3). We, sincerely, believe that for other values of the parameters, it should work as long as moments do exist. Also, we hope that one can use the Bayesian approach and compare it with the MOM and MLE techniques used in this paper, which we intend to take into account in our future research. It is hoped that the findings of the paper will be useful for researchers in the fields of reliability, probability, statistics, and other applied sciences.

ACKNOWLEDGEMENTS

We are thankful to referee for his valuable comments and suggestions, which certainly improved the presentation of the paper. Also, the first author, M. Shakil, is grateful to Miami Dade College for giving him the opportunity to be of service to this institution, without which it would have been impossible to conduct his research. This article was partially completed while the second author, B. M. Golam Kibria, was on sabbatical leave (Fall 2017). He is grateful to Florida International University for awarding him the sabbatical leave which gave him excellent research facilities.

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