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A Generalized Class of Varying Kernel Regression Estimators

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Abstract: Nadaraya-Watson (NW) estimator with fixed bandwidth and its adaptive forms with varying bandwidths are widely used kernel regression estimators in nonparametric regression. In this paper, we propose a generalized class of varying kernel regression estimators with its members based on various statistical measures of pilot density estimates. We study the performance of the members of this class in terms of mean integrated squared error (MISE).

Keywords: Nonparametric Regression, Kernel Regression Estimator, Statistical Measures, Varying Bandwidth, Pilot Density Estimates, Mean Integrated Squared Error (MISE)

1. INTRODUCTION

Nonparametric regression estimation is an important branch of statistical inference as it explores the relation between response variable (Y) and predictor variable (X) in the form of regression function $m(x) = E(Y|X = x)$ from the data with the assumption that both variables are continuous.

Suppose $(X_1, Y_1), \cdots, (X_n, Y_n)$, are n pairs of independent random variables with joint density function $f(x, y)$, the regression model is given by

$$y_i = m(x_i) + \epsilon_i\,,\ i = 1,2,\cdots,n \qquad (1)$$

with $m(.)$ as unknown regression function and observational error ϵ_i being a random variable with mean 0 and variance σ^2.

The earliest kernel estimator to estimate $m(.)$ is due to Nadaraya-Watson (1964) [2], [3] given by

$$\widehat{m}_{NW}(x) = \frac{\frac{1}{n}\sum_{i=1}^{n} Y_i K_h(x-X_i)}{\frac{1}{n}\sum_{i=1}^{n} K_h(x-X_i)} \qquad (2)$$

where $K_h(x - X_i) = \frac{1}{h}K\left(\frac{x-X_i}{h}\right)$, h is fixed bandwidth, $K(.)$ is a kernel function, X_i is ordered observation of X and Y_i is ordered observation of Y corresponding to X_i.

Abramson (1982) [4] established that, the class of modified NW estimators using varying bandwidths $h_* = h\lambda$, $\lambda = \left(\frac{\tilde{f}(x)}{g\left(\tilde{f}(x)\right)}\right)^{-\delta}$ instead of h in (2) has its best performance when sensitivity parameter $\delta = 0.5$ and $g\left(\tilde{f}(x)\right)$ is function of pilot density estimates, $\tilde{f}(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - X_i)$. Studies on members of this class are carried out by Silverman (1986) [1], Demir and Toktamiş (2010) [6], Aljuhani and Al Turk (2014) [5] and Joshi and Deshpande (2016) [7].

In this paper, we propose a generalized class of varying kernel regression estimators in Section 2. In Section 3 and Section 4 respectively, we furnish MISE values of estimators based on location and variation measures. We make a record of observations on performance of the proposed class of estimators in Section 5.

2. GENERALIZED CLASS OF ESTIMATORS

In this section, we propose a generalized class of estimators based on varying NW estimator. The proposed class of estimators is given by

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$$\widehat{m}_{SM_{(\alpha)}}(x) = \frac{\sum_{i=1}^{n} Y_i K_{h_*}(x-X_i)}{\sum_{i=1}^{n} K_{h_*}(x-X_i)}, \quad (3)$$

where SM=Statistical Measure of $\tilde{f}(x)$, $\lambda = \left(\frac{\tilde{f}(x)}{g\left(\tilde{f}(x)\right)}\right)^{-0.5}$, $g\left(\tilde{f}(x)\right) = \frac{SM}{\alpha}$ and $\alpha = (ml)^k$, $m, l, k > 0$.

Here, in expression λ, the sensitivity parameter δ is taken as 0.5, since Abramson(1982) theorize the optimality of δ to be 0.5 for estimators based on varying bandwidths and prior works on study of such estimators have adopted this value of δ.

We consider some statistical measures of pilot density estimates to obtain varying bandwidths. The location measures under consideration are geometric mean $\left(\{\prod_{i=1}^{n} \tilde{f}(x_i)\}^{\frac{1}{n}}, \mathrm{GM}\right)$, arithmetic mean $\left(\frac{\sum_{i=1}^{n}\tilde{f}(x_i)}{n}, AM\right)$, midrange $\left(\frac{\tilde{f}(x_{(n)})+\tilde{f}(x_{(1)})}{2}, MR\right)$, median $\left(\left(\frac{n+1}{2}\right)^{th}\right.$ value of $\tilde{f}(x)$, if n is odd, $\left(\frac{(n/2)+((n/2)+1)}{2}\right)^{th}$ value of $\tilde{f}(x)$, if n is even, $\left.M\right)$ and variance measures under consideration are range $\left(\tilde{f}(x_{(n)}) - \tilde{f}(x_{(1)}), R\right)$, standard deviation $\left(\left(\frac{1}{n}\sum_{i=1}^{n}(\tilde{f}(x_i) - \overline{\tilde{f}(x)})^2\right)^{0.5}, SD\right)$, mean deviation from mean $\left(\frac{1}{n}\sum_{i=1}^{n}|(\tilde{f}(x_i) - \overline{\tilde{f}(x)})|, MD_a\right)$ and mean deviation from median $\left(\frac{1}{n}\sum_{i=1}^{n}|(\tilde{f}(x_i) - \widetilde{\tilde{f}(x)}|, MD_m\right)$ where $\overline{\tilde{f}(x)}$ is mean of $\tilde{f}(.)$, $\widetilde{\tilde{f}(x)}$ is median of $\tilde{f}(.)$ and $\tilde{f}\left(x_{(i)}\right)$ is i^{th} order statistic of $\tilde{f}(.)$.

For different values of m, l and k, we get different members of the proposed generalized class of estimators.

For example, for $m = l = k = 1$, $SM = GM$, the $\widehat{m}_{GM_{(1)}}$ is estimator due to Silverman (1986) [1].

For $m = l = k = 1$, $SM = AM$, the $\widehat{m}_{AM_{(1)}}$ is estimator due to Demir and Toktamiş (2010) [6].

For $m = l = k = 1$, $SM = R$, the $\widehat{m}_{R_{(1)}}$ is estimator due to Aljuhani and Al Turk (2014) [5].

For $m = l = 1, k = 3$, $SM = R$, the $\widehat{m}_{R_{(n^3)}}$ is estimator due to Joshi and Deshpande (2016) [7] and we get more members of the class for various SM and different values of m, l and k

TABLE 1. VALUES OF m, l AND k RESULTING IN VARIOUS VALUES OF α.

m	1	1	1	2	1	1	2	3	2	1	2	3	1	1	$\sqrt{2}$	$\sqrt{3}$	$\sqrt[3]{2}$	1	1
l	1	2	3	2	2	3	2	9	2	n	n	n	n	n	n	n	n	n	n
k	1	1	1	1	3	2	2	1	3	1	1	1	2	3	2	2	3	4	5
α	1	2	3	4	8	9	16	27	64	n	$2n$	$3n$	n^2	n^3	$2n^2$	$3n^2$	$2n^3$	n^4	n^5

Numerous members of the proposed class are obtained using various values of α as listed in table 1 and various statistical measures of $\tilde{f}(x)$. For our study we classify the members of the general class as class of estimators based on location (GM, AM, MR and M) and variance (R, SD, MD_a and MD_m) measures.

MISE [9] of any estimator $\widehat{m}_E$ is the mean of ISE given by

$$ISE(\widehat{m}_E(x)) = \int_0^1 (\widehat{m}_E(x) - m(x))^2 dx \quad (4)$$

We consider the regression function $y = 1 - x + exp(-200 * (x - 0.5)^\wedge 2) + \epsilon$ with $\epsilon \sim N(0,0.01)$ [8] for analyzing the performance of proposed class of estimators. We compute MISE of members of class of estimators under Gaussian kernel, $\frac{1}{\sqrt{2\pi}} exp\{(-u^2/2)\}$, where $u = \frac{x-X_i}{h}$ for NW estimator and $u = \frac{x-X_i}{h_*}$ for the members of the generalized class. MISE is computed using Monte-Carlo simulation with 10,000 repetitions. MISE values of $\widehat{m}_{NW}$ are given in table 2 for various sample sizes.

TABLE 2. MISE VALUES OF $\widehat{m}_{NW}$ FOR VARIOUS SAMPLE SIZES.

n	10	25	50	75	100	150	200	250
MISE of $\widehat{m}_{NW}$	0.0692	0.0546	0.0421	0.0349	0.0300	0.0241	0.0203	0.0175

3. MISE OF ESTIMATORS BASED ON LOCATION MEASURES

In this section, we study the performance of estimators based on location measures, viz. $\widehat{m}_{GM_{(\alpha)}}$, $\widehat{m}_{AM_{(\alpha)}}$, $\widehat{m}_{MR_{(\alpha)}}$ and $\widehat{m}_{M_{(\alpha)}}$ whose varying bandwidths are respectively based on geometric mean, arithmetic mean, midrange and median for various values of m, l and k. Tables 3-6 respectively represent the MISE values of $\widehat{m}_{GM_{(\alpha)}}$, $\widehat{m}_{AM_{(\alpha)}}$, $\widehat{m}_{MR_{(\alpha)}}$ and $\widehat{m}_{M_{(\alpha)}}$ for various values of α.

TABLE 3. MISE VALUES OF MEMBERS OF $\widehat{m}_{GM_{(\alpha)}}$ FOR DIFFERENT VALUES OF α.

n	α									
	1	**2**	**3**	**4**	**8**	**9**	**16**	**27**	**64**	**n**
10	0.0726	0.0515	0.0403	0.0333	0.0204	0.0187	0.0119	0.0077	0.0038	0.0173
25	0.0568	0.0396	0.0312	0.0261	0.0165	0.0153	0.0104	0.0075	0.0046	0.0078
50	0.0425	0.0292	0.0230	0.0193	0.0127	0.0119	0.0090	0.0072	0.0053	0.0058
75	0.0348	0.0236	0.0186	0.0157	0.0111	0.0105	0.0085	0.0072	0.0057	0.0054
100	0.0295	0.0199	0.0159	0.0137	0.0102	0.0098	0.0083	0.0073	0.0059	0.0053
150	0.0238	0.0163	0.0134	0.0119	0.0096	0.0094	0.0084	0.0077	0.0065	0.0053
200	0.0201	0.0142	0.012	0.0109	0.0094	0.0092	0.0084	0.0078	0.0068	0.0052
250	0.0177	0.0129	0.0113	0.0105	0.0093	0.0091	0.0085	0.0080	0.0070	0.0052
$\boldsymbol{n}$	$\boldsymbol{\alpha}$									
	$\boldsymbol{2n}$	$\boldsymbol{3n}$	$\boldsymbol{n^2}$	$\boldsymbol{2n^2}$	$\boldsymbol{3n^2}$	$\boldsymbol{n^3}$	$\boldsymbol{2n^3}$	$\boldsymbol{3n^3}$	$\boldsymbol{n^4}$	$\boldsymbol{n^5}$
10	0.0102	0.0072	0.0027	0.0017	0.0014	0.0007	0.0005	0.0004	2.2 X10^{-04}	7.0 X10^{-05}
25	0.0052	0.0042	0.0015	0.0011	0.0009	0.0003	0.0002	0.0002	6.2 X10^{-05}	1.3 X10^{-05}
50	0.0044	0.0038	0.0012	0.0008	0.0007	0.0002	1.2 X10^{-04}	9.9 X10^{-05}	2.4 X10^{-05}	3.5 X10^{-06}
75	0.0044	0.0038	0.001	0.0007	5.9 X10^{-04}	1.2 X10^{-04}	8.9 X10^{-05}	7.3 X10^{-05}	1.4 X10^{-05}	1.5 X10^{-06}
100	0.0043	0.0038	0.0008	0.0006	5.1 X10^{-04}	9.0 X10^{-05}	6.3 X10^{-05}	5.2 X10^{-05}	8.7 X10^{-06}	9.8 X10^{-07}
150	0.0043	0.0037	0.0007	0.0005	4.2 X10^{-04}	6.2 X10^{-05}	4.3 X10^{-05}	3.5 X10^{-05}	5.1 X10^{-06}	4.0 X10^{-07}
200	0.0043	0.0037	0.0006	4.4 X10^{-04}	3.6 X10^{-04}	4.5 X10^{-05}	3.2 X10^{-05}	2.6 X10^{-05}	3.1 X10^{-06}	2.2 X10^{-07}
250	0.0043	0.0037	0.0005	3.9 X10^{-04}	3.2 X10^{-04}	3.6 X10^{-05}	2.5 X10^{-05}	2.0 X10^{-05}	2.4 X10^{-06}	1.6 X10^{-07}

TABLE 4. MISE VALUES OF MEMBERS OF $\widehat{m}_{AM_{(\alpha)}}$ FOR DIFFERENT VALUES OF α.

$\boldsymbol{n}$	$\boldsymbol{\alpha}$									
	1	**2**	**3**	**4**	**8**	**9**	**16**	**27**	**64**	**n**
10	0.0739	0.0527	0.0414	0.0343	0.021	0.0192	0.0122	0.0078	0.0038	0.0177
25	0.0577	0.0404	0.0319	0.0267	0.0169	0.0156	0.0106	0.0076	0.0047	0.008
50	0.0433	0.0297	0.0234	0.0196	0.0129	0.0121	0.0091	0.0073	0.0053	0.0058
75	0.0353	0.024	0.0189	0.016	0.0112	0.0107	0.0086	0.0073	0.0057	0.0055
100	0.0301	0.0204	0.0162	0.014	0.0104	0.0099	0.0084	0.0074	0.006	0.0053
150	0.024	0.0164	0.0135	0.012	0.0097	0.0094	0.0084	0.0077	0.0065	0.0053
200	0.0203	0.0142	0.0121	0.011	0.0094	0.0092	0.0085	0.0079	0.0068	0.0052
250	0.0174	0.0126	0.011	0.0102	0.009	0.0089	0.0083	0.0078	0.0068	0.0051
n	α									
	$\boldsymbol{2n}$	$\boldsymbol{3n}$	$\boldsymbol{n^2}$	$\boldsymbol{2n^2}$	$\boldsymbol{3n^2}$	$\boldsymbol{n^3}$	$\boldsymbol{2n^3}$	$\boldsymbol{3n^3}$	$\boldsymbol{n^4}$	$\boldsymbol{n^5}$
10	0.01	0.0071	0.0028	0.0017	0.0013	7.1X10^{-04}	4.7X10^{-04}	3.8X10^{-04}	2.2X10^{-04}	7.3X10^{-05}
25	0.0054	0.0044	0.0016	0.0012	0.001	3.3X10^{-04}	2.5X10^{-04}	2.1X10^{-04}	6.7X10^{-05}	1.3X10^{-05}
50	0.0045	0.0039	0.0012	8.6X10^{-04}	7.1X10^{-04}	1.8X10^{-04}	1.3X10^{-04}	1.1X10^{-04}	2.7X10^{-05}	4.1X10^{-06}
75	0.0044	0.0038	0.001	7.2X10^{-04}	5.9X10^{-04}	1.2X10^{-04}	8.8X10^{-05}	7.2X10^{-05}	1.4X10^{-05}	1.7X10^{-06}
100	0.0044	0.0038	8.6X10^{-04}	6.3X10^{-04}	5.2X10^{-04}	9.2X10^{-05}	6.5X10^{-05}	5.4X10^{-05}	1.1X10^{-05}	9.7X10^{-07}
150	0.0043	0.0038	7.1X10^{-04}	5.1X10^{-04}	4.2X10^{-04}	6.0X10^{-05}	4.2X10^{-05}	3.4X10^{-05}	5.0X10^{-06}	3.4X10^{-07}
200	0.0024	0.0021	6.2X10^{-04}	2.5X10^{-04}	2.1X10^{-04}	4.6X10^{-05}	1.9X10^{-05}	1.5X10^{-05}	3.3X10^{-06}	1.4X10^{-07}
250	0.0033	0.0029	5.4X10^{-04}	3.1X10^{-04}	2.5X10^{-04}	3.6X10^{-05}	2.0X10^{-05}	1.6X10^{-05}	2.2X10^{-06}	1.1X10^{-07}

TABLE 5. MISE VALUES OF MEMBERS OF $\hat{m}_{MR_{(\alpha)}}$ FOR DIFFERENT VALUES OF α.

n	α									
	1	**2**	**3**	**4**	**8**	**9**	**16**	**27**	**64**	**n**
10	0.0706	0.0495	0.0385	0.0318	0.0194	0.0177	0.0112	0.0072	0.0035	0.0163
25	0.0546	0.0377	0.0295	0.0246	0.0155	0.0143	0.0098	0.0071	0.0044	0.0074
50	0.0404	0.0275	0.0215	0.0180	0.0120	0.0113	0.0085	0.0069	0.0051	0.0056
75	0.0326	0.0219	0.0173	0.0147	0.0105	0.0100	0.0082	0.0070	0.0055	0.0053
100	0.0279	0.0188	0.0150	0.0130	0.0099	0.0095	0.0081	0.0072	0.0058	0.0052
150	0.0220	0.0151	0.0126	0.0113	0.0093	0.0091	0.0082	0.0075	0.0063	0.0051
200	0.0188	0.0134	0.0115	0.0106	0.0092	0.0090	0.0083	0.0077	0.0067	0.0051
250	0.0166	0.0123	0.0109	0.0102	0.0091	0.0090	0.0084	0.0079	0.0069	0.0051
n	α									
	$2n$	$3n$	n^2	$2n^2$	$3n^2$	n^3	$2n^3$	$3n^3$	n^4	n^5
10	0.0094	0.0066	0.0026	0.0017	0.0013	6.6X10^{-04}	4.8X10^{-04}	3.9X10^{-04}	2.1X10^{-04}	6.4X10^{-05}
25	0.0050	0.0041	0.0015	0.0011	8.7X10^{-04}	3.2X10^{-04}	2.1X10^{-04}	1.7X10^{-04}	6.4X10^{-05}	1.6X10^{-05}
50	0.0043	0.0037	0.0011	8.0X10^{-04}	6.6X10^{-04}	1.7X10^{-04}	1.2X10^{-04}	9.8X10^{-05}	2.3X10^{-05}	3.3X10^{-06}
75	0.0042	0.0036	9.3X10^{-04}	6.7X10^{-04}	5.5X10^{-04}	1.1X10^{-04}	8.1X10^{-05}	6.6X10^{-05}	1.2X10^{-05}	1.5X10^{-06}
100	0.0042	0.0037	8.2X10^{-04}	5.9X10^{-04}	4.8X10^{-04}	8.8X10^{-05}	5.9X10^{-05}	4.8X10^{-05}	8.0X10^{-06}	7.0X10^{-07}
150	0.0041	0.0036	6.7X10^{-04}	4.8X10^{-04}	4.0X10^{-04}	5.6X10^{-05}	4.0X10^{-05}	3.3X10^{-05}	5.0X10^{-06}	3.6X10^{-07}
200	0.0041	0.0036	5.8X10^{-04}	4.2X10^{-04}	3.5X10^{-04}	4.3X10^{-05}	3.1X10^{-05}	2.5X10^{-05}	3.0X10^{-06}	1.6X10^{-07}
250	0.0041	0.0036	5.2X10^{-04}	3.7X10^{-04}	3.1X10^{-04}	3.3X10^{-05}	2.5X10^{-05}	2.0X10^{-05}	2.4X10^{-06}	1.7X10^{-07}

TABLE 6. MISE VALUES OF MEMBERS OF $\hat{m}_{M_{(\alpha)}}$ FOR DIFFERENT VALUES OF α.

n	α									
	1	**2**	**3**	**4**	**8**	**9**	**16**	**27**	**64**	**n**
10	0.0759	0.0545	0.0429	0.0356	0.0219	0.0200	0.0127	0.0082	0.0040	0.0185
25	0.0585	0.0411	0.0325	0.0273	0.0174	0.0161	0.0109	0.0078	0.0048	0.0082
50	0.0438	0.0302	0.0238	0.0199	0.0131	0.0123	0.0092	0.0074	0.0053	0.0058
75	0.0359	0.0244	0.0193	0.0163	0.0114	0.0108	0.0087	0.0074	0.0058	0.0055
100	0.0308	0.0208	0.0166	0.0142	0.0105	0.0101	0.0085	0.0075	0.0061	0.0054
150	0.0242	0.0166	0.0136	0.0121	0.0097	0.0094	0.0084	0.0077	0.0065	0.0053
200	0.0205	0.0144	0.0122	0.0111	0.0094	0.0092	0.0085	0.0079	0.0068	0.0053
250	0.0182	0.0132	0.0114	0.0106	0.0093	0.0092	0.0086	0.0080	0.0071	0.0053
n	α									
	$2n$	$3n$	n^2	$2n^2$	$3n^2$	n^3	$2n^3$	$3n^3$	n^4	n^5
10	0.0106	0.0075	0.0029	0.0018	0.0014	7.1X10^{-04}	5.1X10^{-04}	4.2X10^{-04}	2.2X10^{-04}	7.0X10^{-05}
25	0.0055	0.0044	0.0016	0.0011	9.4X10^{-04}	3.3X10^{-04}	2.3X10^{-04}	1.9X10^{-04}	7.3X10^{-05}	1.7X10^{-05}
50	0.0046	0.0039	0.0012	8.7X10^{-04}	7.2X10^{-04}	1.8X10^{-04}	1.3X10^{-04}	1.1X10^{-04}	2.6X10^{-05}	3.5X10^{-06}
75	0.0044	0.0038	0.0010	7.2X10^{-04}	5.9X10^{-04}	1.2X10^{-04}	8.8X10^{-05}	7.2X10^{-05}	1.4X10^{-05}	1.6X10^{-06}
100	0.0044	0.0039	8.8X10^{-04}	6.4X10^{-04}	5.2X10^{-04}	9.1X10^{-05}	6.4X10^{-05}	5.2X10^{-05}	9.9X10^{-06}	8.4X10^{-07}
150	0.0043	0.0038	7.2X10^{-04}	5.2X10^{-04}	4.3X10^{-04}	6.0X10^{-05}	4.4X10^{-05}	3.6X10^{-05}	4.7X10^{-06}	3.1X10^{-07}
200	0.0043	0.0038	6.3X10^{-04}	4.6X10^{-04}	3.8X10^{-04}	4.7X10^{-05}	3.3X10^{-05}	2.7X10^{-05}	3.6X10^{-06}	2.4X10^{-07}
250	0.0043	0.0038	5.7X10^{-04}	4.0X10^{-04}	3.3X10^{-04}	3.7X10^{-05}	2.7X10^{-05}	2.2X10^{-05}	2.3X10^{-06}	2.0X10^{-07}

4. MISE OF ESTIMATORS BASED ON VARIANCE MEASURES

In this section, we furnish MISE values of estimators based on $\widehat{m}_{R(\alpha)}$, $\widehat{m}_{SD(\alpha)}$, $\widehat{m}_{MD_{a(\alpha)}}$ and $\widehat{m}_{MD_{m(\alpha)}}$ whose varying bandwidths are respectively based on range, standard deviation, mean deviation from mean and mean deviation from median for various values of m, l and k. The tables 7-10 respectively represent MISE values of $\widehat{m}_{R(\alpha)}$, $\widehat{m}_{SD(\alpha)}$, $\widehat{m}_{MD_{a(\alpha)}}$ and $\widehat{m}_{MD_{m(\alpha)}}$ for various values of α.

TABLE 7. MISE VALUES OF MEMBERS OF $\widehat{m}_{R(\alpha)}$ FOR DIFFERENT VALUES OF α.

n	α									
	1	2	3	4	8	9	16	27	64	n
10	0.0560	0.0369	0.0279	0.0226	0.0130	0.0118	0.0072	0.0046	0.0024	0.0108
25	0.0489	0.0334	0.0259	0.0214	0.0133	0.0123	0.0084	0.0062	0.0040	0.0065
50	0.0377	0.0254	0.0198	0.0166	0.0112	0.0105	0.0081	0.0066	0.0049	0.0053
75	0.0309	0.0206	0.0163	0.0139	0.0101	0.0096	0.0079	0.0068	0.0053	0.0051
100	0.0264	0.0177	0.0143	0.0124	0.0096	0.0092	0.0080	0.0070	0.0057	0.0050
150	0.0209	0.0145	0.0121	0.0110	0.0092	0.0090	0.0081	0.0074	0.0062	0.0050
200	0.0177	0.0128	0.0111	0.0103	0.0091	0.0089	0.0082	0.0076	0.0065	0.0049
250	0.0157	0.0119	0.0106	0.0100	0.0090	0.0089	0.0083	0.0078	0.0068	0.0049
n	α									
	$2n$	$3n$	n^2	$2n^2$	$3n^2$	n^3	$2n^3$	$3n^3$	n^4	n^5
10	0.0060	0.0043	0.0018	0.0012	9.7×10^{-04}	5.2×10^{-04}	3.6×10^{-04}	2.9×10^{-04}	1.6×10^{-04}	5.0×10^{-05}
25	0.0045	0.0037	0.0014	0.0010	8.0×10^{-04}	2.9×10^{-04}	2.0×10^{-04}	1.7×10^{-04}	5.7×10^{-05}	1.1×10^{-05}
50	0.0041	0.0035	0.0010	7.5×10^{-04}	6.2×10^{-04}	1.6×10^{-04}	1.1×10^{-04}	9.3×10^{-05}	2.2×10^{-05}	2.8×10^{-06}
75	0.0040	0.0035	8.9×10^{-04}	6.4×10^{-04}	5.2×10^{-04}	1.1×10^{-04}	7.5×10^{-05}	6.1×10^{-05}	1.1×10^{-05}	1.2×10^{-06}
100	0.0040	0.0035	7.8×10^{-04}	5.6×10^{-04}	4.6×10^{-04}	8.1×10^{-05}	5.7×10^{-05}	4.6×10^{-05}	8.0×10^{-06}	8.7×10^{-07}
150	0.0040	0.0035	6.4×10^{-04}	4.6×10^{-04}	3.8×10^{-04}	5.5×10^{-05}	4.0×10^{-05}	3.3×10^{-05}	4.6×10^{-06}	4.4×10^{-07}
200	0.0040	0.0035	5.5×10^{-04}	4.0×10^{-04}	3.3×10^{-04}	4.2×10^{-05}	2.9×10^{-05}	2.4×10^{-05}	2.9×10^{-06}	2.3×10^{-07}
250	0.0040	0.0035	4.9×10^{-04}	3.5×10^{-04}	2.9×10^{-04}	3.2×10^{-05}	2.3×10^{-05}	1.9×10^{-05}	2.2×10^{-06}	1.4×10^{-07}

TABLE 8. MISE VALUES OF FOR MEMBERS OF CLASS OF $\widehat{m}_{SD(\alpha)}$ FOR DIFFERENT VALUES OF α.

n	α									
	1	2	3	4	8	9	16	27	64	n
10	0.0293	0.0175	0.0126	0.0098	0.0054	0.0049	0.0031	0.0022	0.0013	0.0045
25	0.0248	0.0155	0.0118	0.0097	0.0064	0.0060	0.0044	0.0034	0.0023	0.0035
50	0.0181	0.0121	0.0098	0.0086	0.0065	0.0063	0.0051	0.0042	0.0030	0.0033
75	0.0147	0.0105	0.0090	0.0082	0.0067	0.0065	0.0055	0.0046	0.0034	0.0032
100	0.0128	0.0098	0.0087	0.0081	0.0068	0.0067	0.0058	0.0050	0.0037	0.0032
150	0.0109	0.0091	0.0084	0.0080	0.0071	0.0069	0.0061	0.0054	0.0041	0.0031
200	0.0101	0.0089	0.0084	0.0081	0.0073	0.0071	0.0064	0.0057	0.0044	0.0030
250	0.0098	0.0089	0.0085	0.0082	0.0075	0.0073	0.0066	0.0059	0.0047	0.0029
n	α									
	$2n$	$3n$	n^2	$2n^2$	$3n^2$	n^3	$2n^3$	$3n^3$	n^4	n^5
10	0.0027	0.0021	0.0010	6.9×10^{-04}	5.6×10^{-04}	3.1×10^{-04}	2.1×10^{-04}	1.7×10^{-04}	1.0×10^{-04}	3.1×10^{-05}
25	0.0025	0.0021	7.7×10^{-04}	5.5×10^{-04}	4.5×10^{-04}	1.6×10^{-04}	1.1×10^{-04}	9.3×10^{-05}	3.3×10^{-05}	5.9×10^{-06}
50	0.0025	0.0021	5.8×10^{-04}	4.2×10^{-04}	3.4×10^{-04}	8.3×10^{-05}	6.0×10^{-05}	4.9×10^{-05}	1.3×10^{-05}	1.8×10^{-06}
75	0.0025	0.0021	4.8×10^{-04}	3.4×10^{-04}	2.8×10^{-04}	5.6×10^{-05}	4.1×10^{-05}	3.4×10^{-05}	6.0×10^{-06}	6.5×10^{-07}
100	0.0024	0.0021	4.1×10^{-04}	2.9×10^{-04}	2.4×10^{-04}	4.0×10^{-05}	3.0×10^{-05}	2.5×10^{-05}	4.5×10^{-06}	3.4×10^{-07}
150	0.0023	0.0020	3.2×10^{-04}	2.3×10^{-04}	1.9×10^{-04}	2.7×10^{-05}	1.9×10^{-05}	1.5×10^{-05}	2.2×10^{-06}	1.1×10^{-07}

200	0.0023	0.0019	2.7X10^{-04}	1.9X10^{-04}	1.6X10^{-04}	2.0X10^{-05}	1.4X10^{-05}	1.1X10^{-05}	1.4X10^{-06}	8.7X10^{-08}
250	0.0022	0.0019	2.4X10^{-04}	1.7X10^{-04}	1.4X10^{-04}	1.5X10^{-05}	1.1X10^{-05}	9.0X10^{-06}	9.3X10^{-07}	6.3X10^{-08}

TABLE 9. MISE VALUES OF MEMBERS OF $\hat{m}_{MD_{a(\alpha)}}$ FOR DIFFERENT VALUES OF α.

n	**α**									
	1	**2**	**3**	**4**	**8**	**9**	**16**	**27**	**64**	**n**
10	0.0248	0.0144	0.0102	0.0080	0.0045	0.0041	0.0026	0.0019	0.0011	0.0037
25	0.0218	0.0136	0.0104	0.0086	0.0057	0.0054	0.0040	0.0031	0.0021	0.0032
50	0.0161	0.0109	0.0089	0.0079	0.0060	0.0058	0.0047	0.0039	0.0028	0.0031
75	0.0131	0.0096	0.0084	0.0077	0.0063	0.0061	0.0051	0.0043	0.0032	0.0030
100	0.0116	0.0091	0.0082	0.0076	0.0065	0.0063	0.0054	0.0046	0.0035	0.0029
150	0.0102	0.0087	0.0081	0.0077	0.0068	0.0066	0.0058	0.0050	0.0038	0.0028
200	0.0097	0.0086	0.0082	0.0078	0.0070	0.0068	0.0061	0.0053	0.0041	0.0027
250	0.0094	0.0086	0.0082	0.0080	0.0072	0.0070	0.0063	0.0056	0.0044	0.0027
n	**α**									
	$2n$	$3n$	n^2	$2n^2$	$3n^2$	n^3	$2n^3$	$3n^3$	n^4	n^5
10	0.0023	0.0018	0.0009	0.0006	4.9X10^{-04}	2.6X10^{-04}	1.8X10^{-04}	1.5X10^{-04}	8.3X10^{-05}	2.7X10^{-05}
25	0.0023	0.0019	0.0007	0.0005	4.0X10^{-04}	1.4X10^{-04}	1.0X10^{-04}	8.4X10^{-05}	3.0X10^{-05}	7.0X10^{-06}
50	0.0023	0.0019	0.0005	0.0004	3.1X10^{-04}	8.0X10^{-05}	5.6X10^{-05}	4.6X10^{-05}	1.0X10^{-05}	1.5X10^{-06}
75	0.0023	0.0019	0.0004	0.0003	2.5X10^{-04}	5.1X10^{-05}	3.6X10^{-05}	3.0X10^{-05}	2.3X10^{-06}	2.9X10^{-07}
100	0.0022	0.0019	0.0004	0.00026	2.2X10^{-04}	3.8X10^{-05}	2.7X10^{-05}	2.2X10^{-05}	4.0X10^{-06}	2.8X10^{-07}
150	0.0021	0.0018	2.9X10^{-04}	2.1X10^{-04}	1.7X10^{-04}	2.4X10^{-05}	1.7X10^{-05}	1.4X10^{-05}	1.9X10^{-06}	1.1X10^{-07}
200	0.0021	0.0017	2.4X10^{-04}	1.7X10^{-04}	1.4X10^{-04}	1.8X10^{-05}	1.3X10^{-05}	1.0X10^{-05}	1.2X10^{-06}	7.8X10^{-08}
250	0.0020	0.0017	2.1X10^{-04}	1.5X10^{-04}	1.2X10^{-04}	1.3X10^{-05}	9.5X10^{-06}	7.8X10^{-06}	7.7X10^{-07}	3.5X10^{-08}

TABLE 10. MISE VALUES OF MEMBERS OF $\hat{m}_{MD_{m(\alpha)}}$ FOR DIFFERENT VALUES OF α.

n	**α**									
	1	**2**	**3**	**4**	**8**	**9**	**16**	**27**	**64**	**n**
10	0.0238	0.0138	0.0098	0.0076	0.0043	0.0039	0.0025	0.0018	0.0011	0.0036
25	0.0212	0.0132	0.0101	0.0084	0.0056	0.0053	0.0039	0.0030	0.0020	0.0032
50	0.0157	0.0106	0.0088	0.0078	0.0060	0.0057	0.0047	0.0038	0.0027	0.0030
75	0.0128	0.0095	0.0083	0.0076	0.0062	0.0060	0.0051	0.0043	0.0031	0.0030
100	0.0114	0.0090	0.0081	0.0076	0.0064	0.0062	0.0053	0.0046	0.0034	0.0029
150	0.0101	0.0087	0.0081	0.0077	0.0067	0.0065	0.0057	0.0050	0.0038	0.0028
200	0.0096	0.0086	0.0081	0.0078	0.0070	0.0068	0.0060	0.0053	0.0041	0.0027
250	0.0094	0.0086	0.0082	0.0079	0.0071	0.0070	0.0063	0.0055	0.0043	0.0027
n	**α**									
	$2n$	$3n$	n^2	$2n^2$	$3n^2$	n^3	$2n^3$	$3n^3$	n^4	n^5
10	0.0022	0.0017	8.5X10^{-04}	5.9X10^{-04}	4.7X10^{-04}	2.5X10^{-04}	1.8X10^{-04}	1.4X10^{-04}	8.0X10^{-05}	2.6X10^{-05}
25	0.0023	0.0019	6.8X10^{-04}	4.9X10^{-04}	4.0X10^{-04}	1.4X10^{-04}	1.0X10^{-04}	8.2X10^{-05}	2.9X10^{-05}	5.2X10^{-06}
50	0.0023	0.0019	5.2X10^{-04}	3.8X10^{-04}	3.1X10^{-04}	7.8X10^{-05}	5.5X10^{-05}	4.5X10^{-05}	1.2X10^{-05}	1.7X10^{-06}
75	0.0022	0.0019	4.2X10^{-04}	3.0X10^{-04}	2.5X10^{-04}	5.0X10^{-05}	3.6X10^{-05}	3.0X10^{-05}	1.1X10^{-05}	1.7X10^{-06}
100	0.0022	0.0018	3.6X10^{-04}	2.6X10^{-04}	2.1X10^{-04}	3.7X10^{-05}	2.7X10^{-05}	2.2X10^{-05}	3.5X10^{-06}	2.5X10^{-07}
150	0.0021	0.0018	2.8X10^{-04}	2.0X10^{-04}	1.7X10^{-04}	2.3X10^{-05}	1.7X10^{-05}	1.4X10^{-05}	1.9X10^{-06}	1.9X10^{-07}

200	0.0020	0.0017	2.4X10^{-04}	1.7X10^{-04}	1.4X10^{-04}	1.8X10^{-05}	1.2X10^{-05}	9.7X10^{-06}	1.1X10^{-06}	6.3X10^{-08}
250	0.0020	0.0017	2.1X10^{-04}	1.5X10^{-04}	1.2X10^{-04}	1.3X10^{-05}	9.2X10^{-06}	7.5X10^{-06}	8.1X10^{-07}	1.5X10^{-08}

5. OBSERVATIONS ON PERFORMANCE OF PROPOSED CLASS

It is observed from tables that, the members of proposed generalized class of estimators based on location and variation measures outperform NW estimator.

From Tables 3-6, we notice that, the MISE values of members of classes of estimators based on location measures, $\hat{m}_{GM(\alpha)}$, $\hat{m}_{AM(\alpha)}$, $\hat{m}_{MR(\alpha)}$ and $\hat{m}_{M(\alpha)}$ decrease as sample size increase. The MISE values of $\hat{m}_{GM(\alpha)}$ are lesser than $\hat{m}_{AM(\alpha)}$ for smaller sample sizes for $\alpha \leq 27$. The members of subclasses of $\hat{m}_{AM(\alpha)}$ and $\hat{m}_{MR(\alpha)}$ possess least MISE compared to members of $\hat{m}_{M(\alpha)}$ for all the sample sizes. It reveals that the subclass of kernel regression estimators based on AM and MR are close competitors among classes of estimators based on location measures.

From tables 7-10 we observe that, the MISE values of members of subclasses of $\hat{m}_{R(\alpha)}$, $\hat{m}_{SD(\alpha)}$, $\hat{m}_{MD_a(\alpha)}$ and $\hat{m}_{MD_m(\alpha)}$ decrease as sample size increase. Also, MISE values of estimators based on MD_a and MD_m are the least when compared to other members for all the sample sizes, implying that these two classes of estimators outperform the other two classes of estimators among estimators based on variance measures.

We plot the regression estimates obtained using $\hat{m}_{AM(\alpha)}$, $\hat{m}_{MR(\alpha)}$, $\hat{m}_{MD_a(\alpha)}$ and $\hat{m}_{MD_m(\alpha)}$ for some values of α and n in figures 1-4.

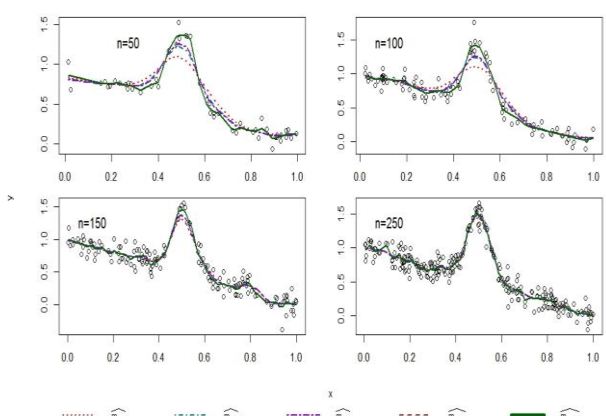

Figure 1. Regression fits based on $\hat{m}_{AM(2)}$, $\hat{m}_{MR(2)}$, $\hat{m}_{MD_a(2)}$ and $\hat{m}_{MD_m(2)}$.

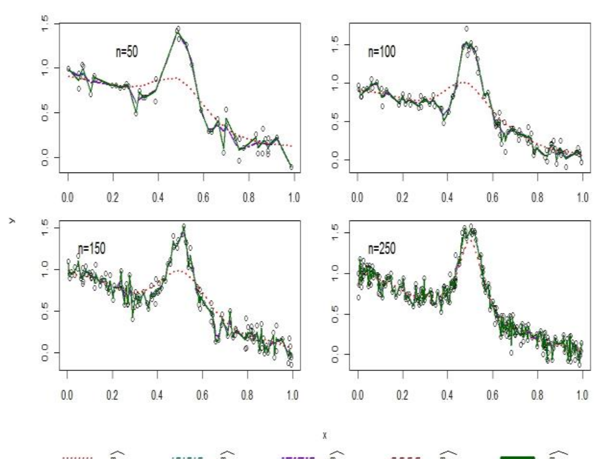

Figure 2. Regression fits based on $\hat{m}_{AM(n)}$, $\hat{m}_{MR(n)}$, $\hat{m}_{MD_a(n)}$ and $\hat{m}_{MD_m(n)}$.

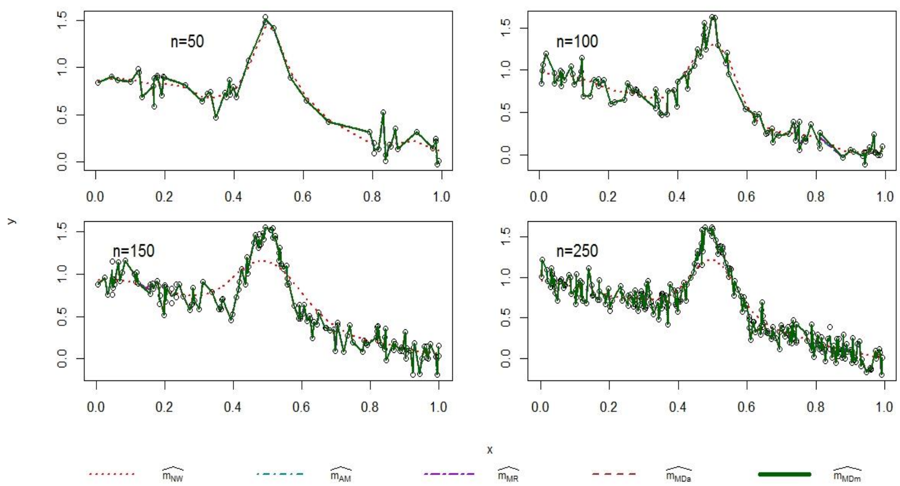

Figure 3. Regression fits based on $\hat{m}_{AM(n^2)}$, $\hat{m}_{MR(n^2)}$, $\hat{m}_{MD_a(n^2)}$ and $\hat{m}_{MD_m(n^2)}$

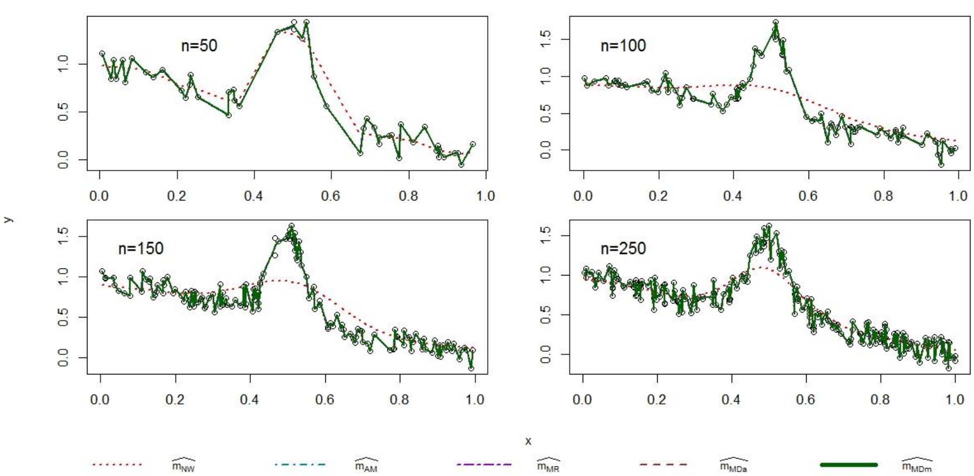

Figure 4. Regression fits based on $\hat{m}_{AM(n^3)}$, $\hat{m}_{MR(n^3)}$, $\hat{m}_{MD_a(n^3)}$ and $\hat{m}_{MD_m(n^3)}$.

From figures 1-4, it is observed that, among the members of subclasses of $\hat{m}_{AM(\alpha)}$, $\hat{m}_{MR(\alpha)}$, $\hat{m}_{MD_a(\alpha)}$ and $\hat{m}_{MD_m(\alpha)}$ the members of subclass of $\hat{m}_{MD_m(\alpha)}$ fits the data well. This reveals that the estimator $\hat{m}_{MD_m(\alpha)}$ outperforms every other subclass of the generalized classes of estimators.

It is also observed that, for $\alpha \geq n$, all subclasses of estimators have MISE values less than 10^{-4}, which may sometimes lead to over smoothing of regression fits. Hence depending on the required level of smoothing, one may consider any member of the proposed generalized class of estimators based on the value of α to obtain optimal results.

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