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Some Characterizations of Chaudhry and Zubair's Extended Generalized Inverse Gaussian Distribution

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Abstract: The objective of this paper is to characterize Chaudhry and Zubair's extended generalized inverse Gaussian distribution [11] [Chaudhry, M. A., Zubair, S. M.: On A Class of Incomplete Gamma Functions, with Applications. Chapman & Hall/CRC, Boca Raton (2002)]. Some distributional properties of the extended generalized inverse Gaussian distribution are given. Based on these distributional properties, several characterizations of the extended generalized inverse Gaussian distribution are given by truncated moment, order statistics and upper record values.

Keywords: Characterization, Extended Generalized Inverse Gaussian Distribution, Order Statistics, Truncated Moment, Upper Record Values

1. Introduction

The generalized inverse Gaussian (GIG) distribution has received special attention in view of its wide applications in many areas of research such as actuaries, biomedicine, demography, environmental and ecological sciences, finance, lifetime data, reliability theory, and traffic data, among others. It was first introduced by Good [17], and later studied by many authors and researchers, such as Sichel [30], Wise [33], Barndorff-Nielsen [9], Jorgensen [20], Iyengar and Liao [18], Seshadri and Wesolowski [28], Wesolowski [32], Chaudhry and Zubair [10, 11], Chou and Huang [12], and Lemonte and Cordeiro [24], among others. For detailed discussions on GIG distribution, the interested readers are referred to Johnson, et al. [19], and Marshall and Olkin [25]. Chaudhry and Zubair [11, p. 266] define the following extended incomplete generalized gamma functions

$$\gamma\left(\alpha, x; b; \beta\right) = \int_0^x x^{\alpha-1} \exp\left(-x - b\,x^{-\beta}\right) dx, \tag{1.1}$$

and

$$\Gamma\left(\alpha, x; b; \beta\right) = \int_x^\infty x^{\alpha-1} \exp\left(-x - b\,x^{-\beta}\right) dx, \tag{1.2}$$

where α, x are complex parameters, $\beta > 0$, and b is a complex variable. By using the definitions (1.1) and (1.2) of the extended incomplete generalized gamma functions, Chaudhry and Zubair [11, p. 266] introduced a new class of four-parameter continuous probability density function (pdf) given by

$$f(x) = \begin{cases} C\left(\alpha, a; b; \beta\right) x^{\alpha-1} \exp\left(-a\,x - b\,x^{-\beta}\right) \\ 0 \end{cases}, \tag{1.3}$$
$$\left(x > 0, a > 0, b \ge 0, \beta > 0, -\infty < \alpha < \infty\right)$$

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and called it the extended generalized inverse Gaussian (EGIG) distribution, where $C = C(\alpha, a; b; \beta) = \left(\int_0^\infty x^{\alpha-1} \exp\left(-a\,x - b\,x^{-\beta} \right) dx \right)^{-1}$ is the normalizing constant. Using the Equations (6.22, 6.31, 6.32, pp. 268 – 269) of Chaudhry and Zubair [11], the normalizing constant is given by

$$C = \left(\frac{1}{a^{\alpha}} \Gamma\left(\alpha, 0; b\,a^{\beta}; \beta\right) \right)^{-1} = \left(\frac{1}{a^{\alpha}} H_{0,2}^{2,0}\left(b\,a^{\beta} \middle| \overline{(0,1)}, \overline{(\alpha, \beta)} \right) \right)^{-1},$$

where $H_{0,2}^{2,0}\left(b \middle| \overline{(0,1)}, \overline{(\alpha, \beta)} \right)$ denotes the Fox H-function; see, for example, Mathai and Saxena [26], among others. When $\beta = 1$, the EGIG reduces to the GIG with the normalizing constant as

$$C = \frac{1}{2}\left(\frac{a}{b} \right)^{\frac{\alpha}{2}} \frac{1}{K_{\alpha}\left(2\sqrt{a\,b} \right)}, \tag{1.4}$$

where $K_{\alpha}\left(2\sqrt{a\,b} \right)$ denotes the modified Bessel function of third kind. It can easily be seen that, by a simple transformation of the variable X or by taking special values of the parameters $\{\alpha, \beta, a, b\}$ in equation (1.3), a number of distributions can be obtained as the special cases of the EGIG distribution; see Chaudhry and Zubair [11]. The corresponding cumulative distribution function (cdf) $F_X(x)$, the reliability function $R(x) = 1 - F_X(x)$ and the hazard functions $h(x) = \frac{f_X(x)}{1 - F_X(x)}$ are respectively given in terms of the extended incomplete generalized incomplete gamma functions as

$$F_X(x) = C\,a^{-\alpha} \gamma\left(\alpha, a\,x; b\,a^{\beta}; \beta\right), \tag{1.5}$$

$$R(x) = C\,a^{-\alpha} \Gamma\left(\alpha, a\,x; b\,a^{\beta}; \beta\right), \tag{1.6}$$

and

$$h(x) = \frac{a^{\alpha} x^{\alpha-1} \exp\left(-a\,x - b\,x^{-\beta} \right)}{\Gamma\left(\alpha, a\,x; b\,a^{\beta}; \beta\right)}. \tag{1.7}$$

The kth moment of the random variable X, for some integer $k > 0$, is given as

$$E\left(X^k\right) = \int_0^\infty x^k f_X(x)\,dx$$

$$= \frac{1}{a^{\alpha+k}} \Gamma\left(\alpha + k, 0; b\,a^{\beta}; \beta\right) = \frac{1}{a^{\alpha+k}} H_{0,2}^{2,0}\left(b\,a^{\beta} \middle| \overline{(0,1)}, \overline{(\alpha + k, \beta)} \right). \tag{1.8}$$

When $k = 1$ in Eq. (1.8), the first moment (or mean) is given as

$$E(X) = \frac{1}{a^{\alpha+1}} \Gamma\left(\alpha + 1, 0; b\,a^{\beta}; \beta\right) = \frac{1}{a^{\alpha+1}} H_{0,2}^{2,0}\left(b\,a^{\beta} \middle| \overline{(0,1)}, \overline{(\alpha + 1, \beta)} \right). \tag{1.9}$$

The possible shapes of the pdf (1.3), cdf (1.5) and hazard function (1.7) of the EGIG are given for some selected values of the parameters in Figures 1.1 to 1.3 respectively. The effects of the parameters can easily be seen from these graphs. For example, it is clear from the plotted Figure 1.1, for selected values of the parameters, that the distributions of the EGIG are positively right skewed with longer and heavier right tails. Also, for the selected values of the parameters, the bathtub shapes of the hazard function of the EGIG are evident from Figure 1.3.

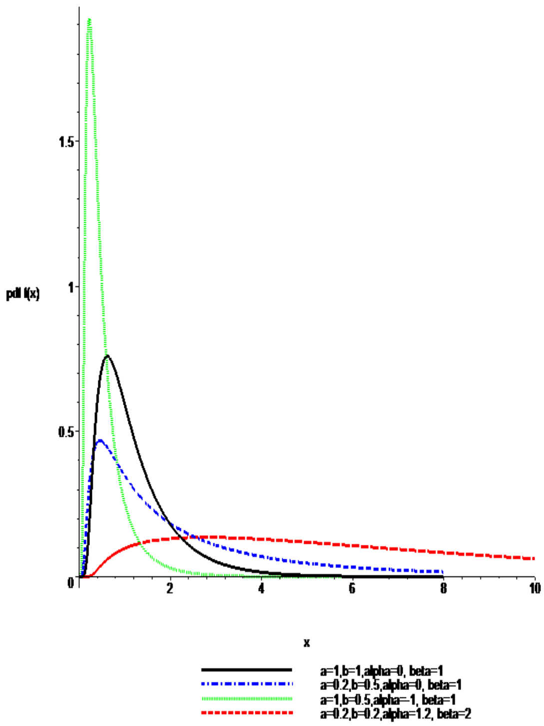

Figure 1.1. PDF Plots of EGIG

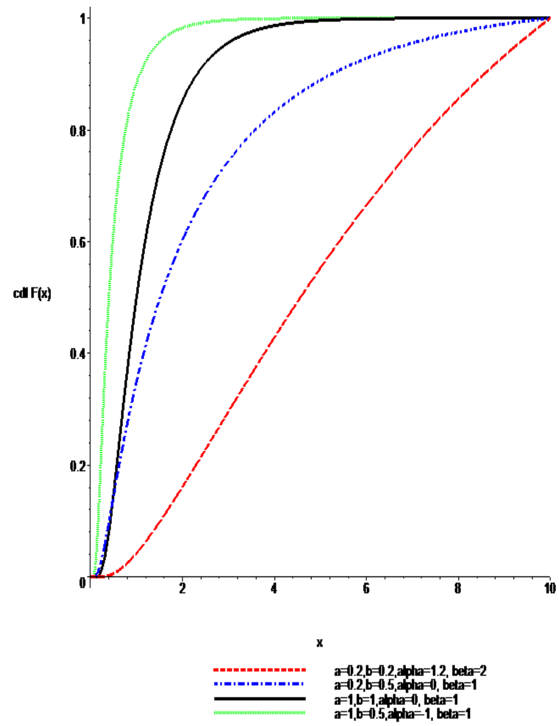

Figure 1.2. CDF Plots of EGIG

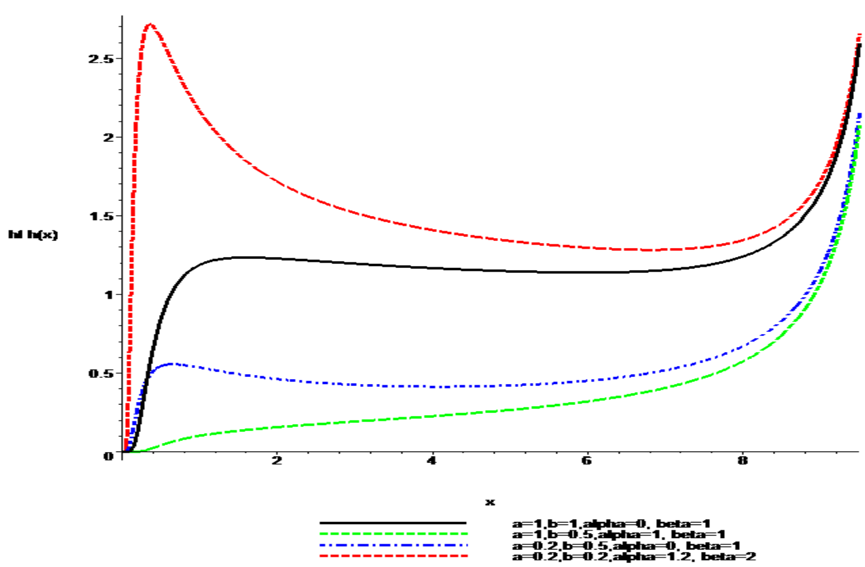

Figure 1.3. Hazard Function Plots of EGIG.

2. CHARACTERIZATIONS

For probability distributions, the characterization problems have been studied by many authors and researchers, see, for example, Ahsanullah [3], Ahsanullah et al. [5, 6, 7], Galambos and Kotz [14], Glänzel [15], Kotz and Shanbhag [21], and Nagaraja [22], and references therein. It is important to characterize a probability distribution subject to certain conditions before it is applied to real world data. According to Nagaraja [22], "A characterization is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model". Similarly, as pointed out by Koudou and Ley [23], "In probability and statistics, a characterization theorem occurs when a given distribution is the only one which satisfies a certain property. Besides their evident mathematical interest per se, characterization theorems also deepen our understanding of the distributions under investigation and sometimes open unexpected paths to innovations which might have been uncovered otherwise". It appears from the literature that many authors and researchers have studied the characterizations of the generalized inverse Gaussian (GIG) distribution. For a very nice survey on the characterizations of GIG, the interested readers are referred to Koudou and Ley [23]. However, to the best knowledge of the authors, no attempt has been made to the characterizations of the extended generalized inverse Gaussian (EGIG) distribution of Chaudhry and Zubair [11]. Motivated by the importance of the characterization problems in the fields of probability and statistics, in this section, we establish our proposed characterization results of the extended generalized inverse Gaussian (EGIG) distribution by truncated moment, order statistics and upper record values. For this, we will need the following assumption and lemmas.

Assumptions 2.1. Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = \inf\{x \mid F(x) > 0\}$, and $\delta = \sup\{x \mid F(x) < 1\}$. We also assume that $f(x)$ is a differentiable for all x, and $E(X)$ exists.

Lemma 2.1. Under the Assumption 2.1, if $E(X \mid X \le x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_0^x \frac{u - g^{/}(u)}{g(u)} du$ is finite for $x > 0$, then

$$f(x) = c e^{\int_0^x \frac{u - g^{/}(u)}{g(u)} du},$$

where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof. For the proof, the interested readers are referred to Shakil, et al. [31].

Lemma 2.2. Under the Assumption 2.1, if $E(X \mid X \ge x) = \tilde{g}(x) r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and $\tilde{g}(x)$ is a continuous differentiable function of x with the condition that $\int_x^\infty \frac{u + \left[\tilde{g}(u)\right]^{/}}{\tilde{g}(u)} du$ is finite for $x > 0$, then

$$f(x) = c e^{-\int_0^x \frac{u + \left[\tilde{g}(u)\right]^{/}}{\tilde{g}(u)} du},$$

where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof. For the proof, the interested readers are referred to Shakil, et al. [31].

2.1. Characterization by Truncated Moment:

Theorem 2.1. If the random variable X satisfies the Assumption 2.1 with $\omega = 0$ and $\delta = \infty$, then

$E\left(X \middle| X \le x\right) = g\left(x\right) \dfrac{f\left(x\right)}{F\left(x\right)}$, where

$$g\left(x\right) = \frac{\gamma\left(\alpha+1, a\,x; b\,a^{\beta}; \beta\right)}{a^{\alpha+1}x^{\alpha-1}\exp\left(-a\,x - b\,x^{-\beta}\right)}, \tag{2.1}$$

if and only if X has the pdf $f_X(x) = C\left(\alpha, a; b; \beta\right) x^{\alpha-1}\exp\left(-a\,x - b\,x^{-\beta}\right)$, where

$$C = C\left(\alpha, a; b; \beta\right) = \left(\frac{1}{a^{\alpha}}\Gamma\left(\alpha, 0; b\,a^{\beta}; \beta\right)\right)^{-1} = \left(\frac{1}{a^{\alpha}} H_{0,2}^{2,0}\left(b\,a^{\beta} \middle| \overline{}, \overline{} \atop (0,1), (\alpha, \beta)\right)\right)^{-1}.$$

Proof. Suppose that $E\left(X \middle| X \le x\right) = g\left(x\right) \dfrac{f\left(x\right)}{F\left(x\right)}$. Then, since $E\left(X \middle| X \le x\right) = \dfrac{\int_0^x u\, f\left(u\right) du}{F\left(x\right)}$, we have $g\left(x\right) = \dfrac{\int_0^x u\, f\left(u\right) du}{f\left(x\right)}$. Now, if the random variable X satisfies the Assumption 2.1 and has the distribution with the pdf (1.3), then we have

$$g\left(x\right) = \frac{\int_0^x u\, f\left(u\right) du}{f\left(x\right)} = \frac{\int_0^x u^{\alpha}\exp\left(-a\,u - b\,u^{-\beta}\right) du}{x^{\alpha-1}\exp\left(-a\,x - b\,x^{-\beta}\right)}.$$

$$= \frac{\gamma\left(\alpha+1, a\,x; b\,a^{\beta}; \beta\right)}{a^{\alpha+1}x^{\alpha-1}\exp\left(-a\,x - b\,x^{-\beta}\right)},$$

using the definition of the extended incomplete generalized incomplete gamma function.

Conversely, suppose that

$$g\left(x\right) = \frac{\gamma\left(\alpha+1, a\,x; b\,a^{\beta}; \beta\right)}{a^{\alpha+1}x^{\alpha-1}\exp\left(-a\,x - b\,x^{-\beta}\right)}. \qquad .$$

Since, by Lemma 2.1, $g'(x) = x - g\left(x\right)\dfrac{f'(x)}{f(x)}$, see Shakil, et al. [31], differentiating $g\left(x\right)$ with respect to x, we have

$$g'(x) = x - g\left(x\right)\left(\frac{\alpha-1}{x} - a + \frac{b\,\beta}{x^{\beta+1}}\right),$$

from which we obtain

$$\frac{x - g'\left(x\right)}{g\left(x\right)} = \frac{\alpha-1}{x} - a + \frac{b\,\beta}{x^{\beta+1}}.$$

Now, since, by Lemma 2.1, we have

$$\frac{x - g'\left(x\right)}{g\left(x\right)} = \frac{f'\left(x\right)}{f\left(x\right)},$$

it follows that

$$\frac{f'(x)}{f(x)} = \frac{\alpha - 1}{x} - a + \frac{b\beta}{x^{\beta+1}}.$$

On integrating the above expression with respect to x and simplifying, we obtain

$$\ln f(x) = \ln\left(c\, x^{\alpha-1} \exp\left(-a x - b x^{-\beta}\right)\right),$$

or,

$$f(x) = c\, x^{\alpha-1} \exp\left(-a x - b x^{-\beta}\right),$$

where c is the normalizing constant to be determined. Thus, on integrating the above equation with respect to x, from $x = 0$ to $x = \infty$, and using the condition $\int_0^\infty f(x)dx = 1$, and the Equations (6.22, 6.31, 6.32, pp. 268 – 269) of Chaudhry and Zubair [11], we obtain $c = \left(\frac{1}{a^\alpha}\Gamma\left(\alpha, 0; b a^\beta; \beta\right)\right)^{-1} = \left(\frac{1}{a^\alpha} H_{0,2}^{2,0}\left(b a^\beta \middle| \overline{(0,1)}, \overline{(\alpha, \beta)}\right)\right)^{-1}$.

This completes the proof.

Theorem 2.2. If the random variable X satisfies the Assumption 2.1 with $\omega = 0$ and $\delta = \infty$, then $E\left(X \middle| X \ge x\right) = \tilde{g}(x)\frac{f(x)}{1 - F(x)}$, where $\tilde{g}(x) = \frac{\left(E(X) - g(x) f(x)\right)\Gamma\left(\alpha, 0; b a^\beta; \beta\right)}{a^\alpha x^{\alpha-1} \exp\left(-a x - b x^{-\beta}\right)}$, where $g(x)$ is given by Eq. (2.1) and $E(X)$ is given by Eq. (1.9), if and only if X has the pdf

$f_X(x) = C(\alpha, a; b; \beta) x^{\alpha-1} \exp\left(-a x - b x^{-\beta}\right)$, where

$$C = C(\alpha, a; b; \beta) = \left(\frac{1}{a^\alpha}\Gamma\left(\alpha, 0; b a^\beta; \beta\right)\right)^{-1} = \left(\frac{1}{a^\alpha} H_{0,2}^{2,0}\left(b a^\beta \middle| \overline{(0,1)}, \overline{(\alpha, \beta)}\right)\right)^{-1}.$$

Proof. Suppose that $E\left(X \middle| X \ge x\right) = \tilde{g}(x)\frac{f(x)}{1 - F(x)}$. Then, since $E\left(X \middle| X \ge x\right) = \frac{\int_x^\infty u f(u) du}{1 - F(x)}$, we have

$\tilde{g}(x) = \frac{\int_x^\infty u f(u) du}{f(x)}$. Now, if the random variable X satisfies the Assumptions 2.1 and has the distribution with the pdf (1.3), then we have

$$\tilde{g}(x) = \frac{\int_x^\infty u f(u) du}{f(x)} = \frac{\int_0^\infty u f(u) du - \int_0^x u f(u) du}{f(x)}$$
$$= \frac{\left(E(X) - g(x) f(x)\right)\Gamma\left(\alpha, 0; b a^\beta; \beta\right)}{a^\alpha x^{\alpha-1} \exp\left(-a x - b x^{-\beta}\right)}.$$

Conversely, suppose that $\tilde{g}(x) = \frac{\left(E(X) - g(x) f(x)\right)\Gamma\left(\alpha, 0; b a^\beta; \beta\right)}{a^\alpha x^{\alpha-1} \exp\left(-a x - b x^{-\beta}\right)}$. Since, by Lemma 2.2,

$\left(\tilde{g}(x)\right)' = -x - \tilde{g}(x)\frac{f'(x)}{f(x)}$, see Shakil, et al. [31], differentiating $\tilde{g}(x)$ with respect to x, we have

$$\left(\tilde{g}(x)\right)^{/} = -x - \tilde{g}(x)\left(\frac{\alpha-1}{x} - a + \frac{b\beta}{x^{\beta+1}}\right),$$

from which we obtain $\dfrac{x+\left(\tilde{g}(x)\right)^{/}}{\tilde{g}(x)} = -\left(\dfrac{\alpha-1}{x} - a + \dfrac{b\beta}{x^{\beta+1}}\right).$

Now, since, by Lemma 2.2, we have

$$\frac{f^{/}(x)}{f(x)} = -\frac{x+\left[\tilde{g}(x)\right]^{/}}{\tilde{g}(x)},$$

it follows that

$$\frac{f^{/}(x)}{f(x)} = \frac{\alpha-1}{x} - a + \frac{b\beta}{x^{\beta+1}}.$$

On integrating the above expression with respect to x and simplifying, we obtain

$$\ln f(x) = \ln\left(c\,x^{\alpha-1}\exp\left(-a\,x - b\,x^{-\beta}\right)\right),$$

or,

$$f(x) = c\,x^{\alpha-1}\exp\left(-a\,x - b\,x^{-\beta}\right),$$

where c is the normalizing constant to be determined. Thus, on integrating the above equation with respect to x, from $x=0$ to $x=\infty$, and using the condition $\int_0^\infty f(x)dx = 1$, we obtain

$c = \left(\dfrac{1}{a^{\alpha}}\Gamma\left(\alpha, 0; b\,a^{\beta}; \beta\right)\right)^{-1} = \left(\dfrac{1}{a^{\alpha}} H_{0,2}^{2,0}\left(b\,a^{\beta}\left|\begin{matrix} \overline{}, & \overline{} \\ (0,1), & (\alpha,\beta)\end{matrix}\right.\right)\right)^{-1}$. This completes the proof.

2.2. Characterizations by Order Statistics: If $X_1, X_2, \ldots, X_n$ be the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and pdf $f(x)$, and if $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ be the corresponding order statistics, then it is known from Ahsanullah, et al. [4], chapter 5, or Arnold, et al. [8], chapter 2, that $X_{j,n} \mid X_{k,n} = x$, for $1 \le k < j \le n$, is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v \mid x)$ where $f_V(v \mid x) = \dfrac{f(v)}{1-F(x)}$, $0 \le v < x$, and $X_{i,n} \mid X_{k,n} = x, 1 \le i < k \le n$, is distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w \mid x)$ where $f_W(w \mid x) = \dfrac{f(w)}{F(x)}$, $w < x$. Let

$S_{k-1} = \frac{1}{k-1}\left(X_{1,n} + X_{2,n} + \ldots + X_{k-1,n}\right)$, and $T_{k,n} = \frac{1}{n-k}\left(X_{k+1,n} + X_{k+2,n} + \ldots + X_{n.n}\right)$.

Theorem 2.3: Suppose the random variable X satisfies the Assumption 2.1 with $\omega = 0$ and $\delta = \infty$, then $E(S_{k-1} \mid X_{k,n} = x) = g(x)\tau(x)$, where $\tau(x) = \dfrac{f(x)}{F(x)}$ and

$$g(x) = \frac{\gamma\left(\alpha+1, a\,x; b\,a^{\beta}; \beta\right)}{a^{\alpha+1}x^{\alpha-1}\exp\left(-a\,x - b\,x^{-\beta}\right)},$$

if and only if X has the pdf $f_X(x) = C(\alpha, a; b; \beta) x^{\alpha-1} \exp\left(-a\,x - b\,x^{-\beta}\right)$, where

$$C = C(\alpha, a; b; \beta) = \left(\frac{1}{a^{\alpha}} \Gamma\left(\alpha, 0; b\,a^{\beta}; \beta\right)\right)^{-1} = \left(\frac{1}{a^{\alpha}} H_{0,2}^{2,0}\left(b\,a^{\beta} \middle| \overline{(0,1),}, \overline{(\alpha, \beta)}\right)\right)^{-1}.$$

Proof: It is known that $E(S_{k-1} \mid X_{k,n} = x) = E(X \mid X \le x)$; see Ahsanullah, et al.[4], and David and Nagaraja [13]. Hence, by Theorem 2.1, the result follows.

Theorem 2.4: Suppose the random variable X satisfies the Assumption 2.1 with $\omega = 0$ and $\delta = \infty$, then

$E(T_{k,n} \mid X_{k,n} = x) = \tilde{g}(x) \dfrac{f(x)}{1 - F(x)}$, where

$$\tilde{g}(x) = \frac{\left(E(X) - g(x) f(x)\right) \Gamma\left(\alpha, 0; b\,a^{\beta}; \beta\right)}{a^{\alpha} x^{\alpha-1} \exp\left(-a\,x - b\,x^{-\beta}\right)},$$

where $g(x)$ is given by Eq. (2.1) and $E(X)$ is given by Eq. (1.9), if and only if X has the pdf

$f_X(x) = C(\alpha, a; b; \beta) x^{\alpha-1} \exp\left(-a\,x - b\,x^{-\beta}\right)$, where

$$C = C(\alpha, a; b; \beta) = \left(\frac{1}{a^{\alpha}} \Gamma\left(\alpha, 0; b\,a^{\beta}; \beta\right)\right)^{-1} = \left(\frac{1}{a^{\alpha}} H_{0,2}^{2,0}\left(b\,a^{\beta} \middle| \overline{(0,1),}, \overline{(\alpha, \beta)}\right)\right)^{-1}.$$

Proof: Since $E(T_{k,n} \mid X_{k,n} = x) = E(X \mid X \ge x)$, see Ahsanullah et al. [4], and David and Nagaraja [13], the result follows from Theorem 2.2.

2.3. Characterization by Upper Record Values: For details on record values, see Ahsanullah [1]. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. If $Y_n = \max(X_1, X_2, \ldots, X_n)$ for $n \ge 1$ and $Y_j > Y_{j-1}$, $j > 1$, then X_j is called an upper record value of $\{X_n, n \ge 1\}$. The indices at which the upper records occur are given by the record times $\{U(n) > \min(j \mid j > U(n+1), X_j > X_{U(n-1)}, n > 1)\}$ and $U(1) = 1$. Let the nth upper record value be denoted by $X(n) = X_{U(n)}$.

Theorem 2.5: Suppose the random variable X satisfies the Assumption 2.1 with $\omega = 0$ and $\delta = \infty$, then

$E(X(n+1) \mid X(n) = x) = \tilde{g}(x) \dfrac{f(x)}{1 - F(x)}$, where

$$\tilde{g}(x) = \frac{\left(E(X) - g(x) f(x)\right) \Gamma\left(\alpha, 0; b\,a^{\beta}; \beta\right)}{a^{\alpha} x^{\alpha-1} \exp\left(-a\,x - b\,x^{-\beta}\right)},$$

where $g(x)$ is given by Eq. (2.1) and $E(X)$ is given by Eq. (1.9), if and only if X has the pdf

$f_X(x) = C(\alpha, a; b; \beta) x^{\alpha-1} \exp\left(-a\,x - b\,x^{-\beta}\right)$, where

$$C = C(\alpha, a; b; \beta) = \left(\frac{1}{a^{\alpha}} \Gamma(\alpha, 0; b\, a^{\beta}; \beta) \right)^{-1} = \left(\frac{1}{a^{\alpha}} H_{0,2}^{2,0}\left(b\, a^{\beta} \middle| \overline{(0,1),}, \overline{(\alpha, \beta)} \right) \right)^{-1}.$$

Proof: It is known from Ahsanullah, et al. [4], and Nevzorov [28] that $E(X(n+1) \mid X(n) = x) = E(X \mid X \geq x)$. Then, the result follows from Theorem 2.2.

3. Concluding Remarks

A characterization of a particular probability distribution states that it is the only distribution that satisfies some specified conditions. According to Glänzel [16], these characterizations may serve as a basis for parameter estimation. Further, Glänzel [16] points out that the characterizations by truncated moments may also be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions. Also, see Glänzel [15]. These conditions are used by various authors to test goodness of fit, efficiency of a particular test of hypothesis and the power of a particular estimating, etc. For example, Volkova and Nikitin [33] used a well-known characterization result of Ahsanullah (see [2]) to test exponentiality of a distribution. For details, see Volkova and Nikitin [33]. For an excellent survey of goodness-of-fit and symmetry tests based on the characterization properties of distributions, the interested readers are referred to two recent nice papers of Nikitin [29] and Miloševic [27], respectively, and references therein. Motivated by these facts, in this paper, we have established some new characterization results of Chaudhry and Zubair's extended generalized inverse Gaussian distribution [11] by left and right truncated moments, order statistics and upper record values in Theorems 2.1 – 2.5. It is hoped that these characterizations may serve as a basis for parameter estimation and in developing some goodness-of-fit tests of Chaudhry and Zubair's extended generalized inverse Gaussian distribution [11]. Also, it is hoped that the findings of this paper will be quite useful for the researchers and practitioners in various fields of theoretical and applied sciences, such as biomedicine, demography, environmental science, ecological science, finance, lifetime, among others.

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