Symbolic Approach for the Applicability of the Third Integral of Motion

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ABSTRACT
In the present paper, literal analytical solutions in recurrent power series forms are developed for the plane motion in an axisymmetric potential to study the applicability of the third integral of motion. For the computational developments of these solutions, efficient method using continued fraction theory together with the device of interval division are provided. Moreover some numerical and graphical illustrations are also given. It was found that, the third integral of motion exists for only a limited range of initial conditions. But it may exist whatever the time interval may be, if it's initial value $\varepsilon(0, 0.12)$.

KEYWORDS: Symbolic approach, axisymmetric potential, third body of motion.

INTRODUCTION
The plane motion in an axisymmetric potential is vital to galactic dynamics, especially for flattened galactic systems, upon such potential the forces and the stars orbits in these systems could be determined (Binney and Tremain 1987). These latter results are extremely important in galactic studies, one of their most explored aspects is the problem of correlation between the parameters of the orbit of a star and its physical properties. Moreover, the motion of an individual star is subject to physical laws of invariance or, to use mathematical name, *integrals of motion*. Two of these conservation laws have been known for long time: conservation of energy and of angular momentum. For long time these two were thought to be the only two. However, certain aspects of individual orbits point at the existence of a third integral of the mechanical properties of the galactic orbits.

The existence of the third integral of motion was usually investigated using numerical schemes. A pioneer paper in this respect was published by He'non and Heiles in1964, (hereafter will be referred to as Paper I), where they carried out the orbit determination numerically using Adams and Runge- Kutta methods.

Undoubtedly, true that, the numerical integration methods can provide very accurate models. But certainly, if full analytical formulae are utilized with nowadays existing symbols used for manipulating digital computer programs, they definitely become invaluable for obtaining models with desired accuracy. Moreover, these analytical formulae usually offer much deeper insight into the nature of a model as compared to numerical integration.

Indeed, in the absence of closed analytical solution of a given differential system, the power series solution (which of course assumed to be convergent) can serve as the analytical representation of its solution. Moreover, it is worth noting that the power series is one of the most powerful methods of mathematical analysis and is no less (and some – times even more) convenient than the elementary functions especially when the problems are to be studied on

computers. In fact, most computers often use series in the calculations of the majority of the elementary functions.

Due to the importance of the plane motion in an axisymmetric potential as mentioned briefly in providing some basic material in studying the existence of the third integral of motion and, on the other hand the importance role of the analytical solution to a problem, the present work is devoted.

In the present paper, literal analytical solutions in recurrent power series forms are developed for the plane motion in an axisymmetric potential to study the applicability of the third integral of motion. For the computational developments of these solutions, efficient method using continued fraction theory together with the device of interval division are provided. Moreover some numerical and graphical illustrations are also given. It was found that, the third integral of motion exists for only a limited range of initial conditions. But it may exist whatever the time interval may be, if it's initial value $\varepsilon(0, 0.12)$.

BASIC EQUATIONS
The Equations of motion of a particle in the plane (x, y) in an arbitrary potential $U(x, y)$ are:

$$\ddot{x} = -\frac{\partial U}{\partial x}; \qquad \ddot{y} = -\frac{\partial U}{\partial y} \tag{1}$$

where 'dot' denotes the differentiation with respect to the time t.

For the present study, the potential H given in Paper I will be used, where

$$H(x, y) = \frac{1}{2}(x^2 + y^2 + 2x^2 y - \frac{2}{3}y^3) \tag{2}$$

because: (1) it is analytically simple; this makes the orbit determination of the trajectory easy; (2) at the same time it is sufficiently complicated to give the trajectories which are far from trivial, as already shown in Paper I. It seems probable that the above potential is a typical representative of the general case, and nothing would be fundamentally changed by addition of higher-order terms.

The equations of motion in the potential H are therefore given as:
$$\ddot{x} = -x - 2xy \tag{3}$$
$$\ddot{y} = -y - x^2 + y^2 \tag{4}$$

Multiplying Equation (3) by $\dot{x}$ and Equation (4) by $\dot{y}$ and then adding, we obtain

$$\frac{1}{2}\left[x^2 + y^2 + \dot{x}^2 + \dot{y}^2\right] + x^2 y - \frac{1}{3}y^3 = h, \tag{5}$$

where h is a constant, so the system of Equations (3) and (4) is a conservative system having the integral of Equation (5)

Let

$$y_1 = x, \qquad y_2 = \dot{x}, \qquad y_3 = y, \qquad y_4 = y$$

Then the system of Equations (3) and (4) and the integral (5) become

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -y_1 - 2y_1 y_3$$

$$\dot{y}_3 = y_4 \qquad\qquad (6)$$

$$\dot{y}_4 = -y_3 - y_1^2 + y_3^2$$

where

$$\frac{1}{2}\left[y_1^2 + y_2^2 + y_3^2 + y_4^2\right] + y_1^2 y_3 - \frac{1}{3}y_3^3 = h \qquad (7)$$

The system of Equations (6) is to be solved subject (say) to the initial conditions

$$\text{at } t = t_0, \qquad y_1 = a, \qquad y_2 = b, \qquad y_3 = c, \qquad y_4 = d \qquad (8)$$

ANALYTICAL SOLUTION

As we mentioned above, in the absence of closed analytical solution of a given differential system the power series solution can serve as the analytical representation of its solution.

The power series solution of the above system could be obtained by assuming the power series for the variables as:

$$y_1 = \sum_{n=1}^{\infty} A_n (t - t_0)^{n-1} \qquad (9)$$

$$y_2 = \sum_{n=1}^{\infty} B_n (t - t_0)^{n-1} \qquad (10)$$

$$y_3 = \sum_{n=1}^{\infty} C_n (t - t_0)^{n-1} \qquad (11)$$

$$y_4 = \sum_{n=1}^{\infty} D_n (t - t_0)^{n-1} \qquad (12)$$

Substituting Equations (9), (10), (11) and (12) into Equations (6) then equating the coefficients of equal powers of $(t - t_0)$ in both sides of the resulting equations we get the recursion equations

$$n\,A_{n+1} = B_n \qquad (13)$$

$$n\,B_{n+1} = -A_n - 2\sum_{j=1}^{n} A_j C_{n-j+1} \qquad (14)$$

$$nC_{n-1} = D_n \qquad (15)$$

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$$n\mathrm{D}_{n+1} = -\mathrm{C}_n - \sum_{j=1}^{n} \mathrm{A}_j \mathrm{A}_{n-j-1} + \sum_{j=1}^{n} \mathrm{C}_j \mathrm{C}_{n-j+1} \tag{16}$$

From Equations (8), (9), (10), (11) and (12) we get

$$\mathrm{A}_1 = a, \qquad \mathrm{B}_1 = b, \qquad \mathrm{C}_1 = c; \qquad \mathrm{D}_1 = d \tag{17}$$

Equations (13), (14), (15) and (16) are applied $\forall\, n = 2,3,...,n_t$, where n_t is the number of terms of the power series. It should also be noted that, if these equations are used in the same order as they stand, then all the coefficients of the power series are completely determined in a full recursive way.

Because of space limitations, only the first ten of $\mathrm{A}'s, \mathrm{B}'s, \mathrm{C}'s$ and $\mathrm{D}'s$ coefficients are listed in Appendix A.

COMPUTATIONAL DEVELOPMENTS

In fact, continued fraction expansions are, generally far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series. Due to the importance of accurate evaluations and the efficiency of continued fractions, I purpose to use them as the computational tools for evaluating the components of the position and velocity vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$. To do so, two steps are to be performed:
1. Transform the given power series into continued fraction.
2. Evaluating the resulting continued fraction.

EULER'S TRANSFORMATION

Generally an infinite series (a power series is special case of it) of functions could be converted into a continued fraction according to Eulers transformation (Battin 1999) which is:

$$\sum_{k=0}^{\infty} U_K \equiv \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{n_3}{d_3 + \cfrac{n_4}{...}}}} \equiv \frac{n_1}{d_1 +}\ \frac{n_2}{d_2 +}\ \frac{n_3}{d_3 +}\ \frac{n_4}{d_4 +}\ + ...$$

where
$$n_1 = U_0\ ;\ n_2 = U_1;\ n_i = -U_{i-1} \times U_{i-3}, \forall i \geq 3$$

$$d_1 = 1\ ;\ d_j = U_{j-2} + U_{j-1} \forall i \geq 2.$$

TOP-DOWN CONTINUED FRACTION EVALUATION

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the nth convergent were accumulated separately with three-term recurrence formulae. The draw back of the first method, obviously, has to decide far down the fraction to being in order to ensure

convergence. The draw back to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm that works from top down while avoiding numerical difficulties would be ideal from a programming standpoint.

Gautschi, (1967) proposed very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as:

$$q = \frac{n_1}{d_1 +} \ \frac{n_2}{d_2 +} \ \frac{n_3}{d_3 +} \ \cdots$$

then initialize the following parameters
$a_1 = 1$, $b_1 = n_1 / d_1$, $q_1 = n_1 / d_1$

and iterate ($k=1,2,\ldots$) according to

$$a_{k-1} = \frac{1}{1 + \dfrac{n_{k+1}}{d_k d_{k+1}} a_k}$$

$$b_{k+1} = (a_k - 1)b_k$$

$$q_{k-1} = q_k + b_{k+1}.$$

In the limit, the q sequence converges to the value of the continued fraction

UTILIZATIONS
When the time interval $(t - t_0)$ is sufficiently large, we may (as usually done for all initial value problems) divide this interval into some intervals each of short length e.g. the interval $[t - t_0]$ may be divided into q intervals $[t_1 - t_0], [t_2 - t_1], \ldots [t - t_{q-1}]$, such that $t_s - t_{s-1} \prec t - t_0$. Then solve the initial value problem for the first interval to find the solution at the time t_1. The solution at t_1 could then be used as the initial conditions for the second interval and so on. By this artifice one needs small numbers of the coefficients for the power series representation in each interval. So that the number of coefficients listed in Appendix A are very sufficient to predict the motion in the interval $(t - t_0)$.

NUMERICAL EXPERIMENTS
Since an integral of motion should be a constant during the motion, so we can adopt for the existence of the third integral of motion h (Equation (5) or (7)) the criterion that:

$$\Delta h = | h - h_0 | \prec \varepsilon \tag{18}$$

where h is value of the integral at the time t, while h_0 its value at the time t_0 and ε is a given tolerance, so using Equation (8), we then get:

$$h_0 = \frac{1}{2}(a^2 + b^2 + c^2 + d^2) + a^2 c - \frac{1}{3}c^3 \tag{19}$$

By using for ε the value 10^{-10} many experiments are performed of them are the following:

1. The third integral of motion exists (in the sense of the criterion mentioned above) for some initial values, of these are those listed in Table 1.

Table 1. Some acceptable initial values for the existence of the third integral of motion and the values of t_0, t, Δh

a	b	c	d	t_0	t	Δh
0.5	0.9	-0.7	0.9	0	5	5.10703×10^{-15}
0.8	1.9	-0.7	0.6	0	1	1.61648×10^{-13}
0.4	0.7	0.5	-0.6	2	5	2.44299×10^{-15}
-0.7	-0.2	-0.5	-0.9	0	2	3.33622×10^{-13}
1.0	1.0	1.0	1.0	0	2	1.66978×10^{-13}

2. Some times the third integral exist for subinterval of $[t - t_0]$ and does not exist for the remaining of the interval, for example: $a = 0.6, b = 0.3, c = 0.6, d = 0.4, t_0 = 0, t = 15$, the integral exist up to $t = 9.35$ with $\Delta h = O(10^{-14})$, then Δh increases very rapidly to the value of 0.325184 at $t=10.8$ and then continually increase to $\approx \infty$.

3. The most important results are that, the third integral of motion may exist for all values of $0 \prec h_0 \prec 0.12$ whatever the interval of time may be. Typical example of this case is: $a = 0.06, b = 0.03, \quad c = 0.06, d = 0.04, \quad t_0 = 0, t = 70 \quad$ with $h_0 = 0.004994$. Graphical representation of this case is shown in Figure 1 (the time axis is $t \times 20$).
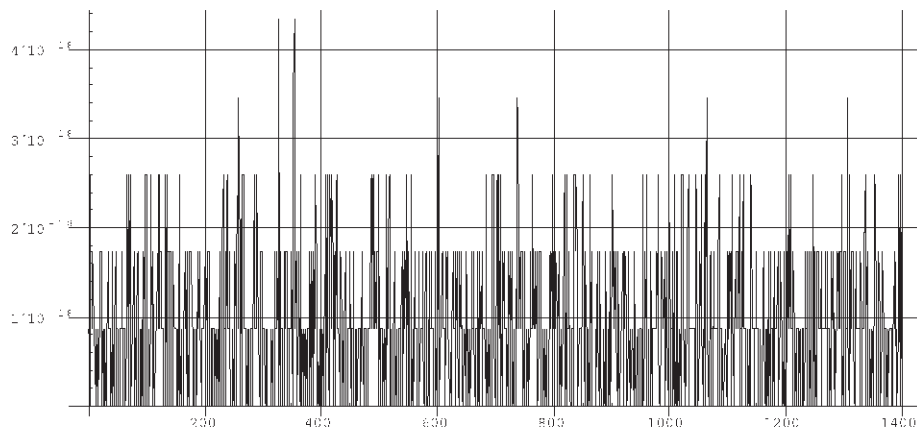

Figure1. Typical behavior for the case in which $0 \prec h_0 \prec 0.12$

4. The graphical representations for the solutions y_j; $j = 1, 2, 3, 4$, for the first case of Table 1 are illustrated in Figure 2.

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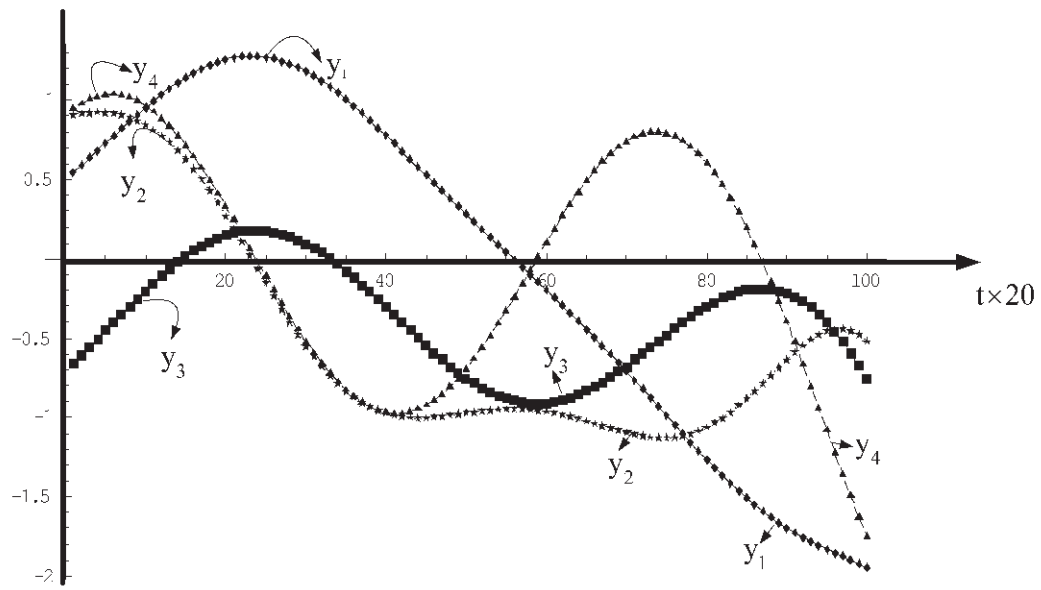Figure 2. The solutions y_j; $j = 1,2,3,4$ for the first case of Table 1

5. The polar plot for a typical case of $0 \prec h_0 \prec 0.12$ are given in Figures 3-6.
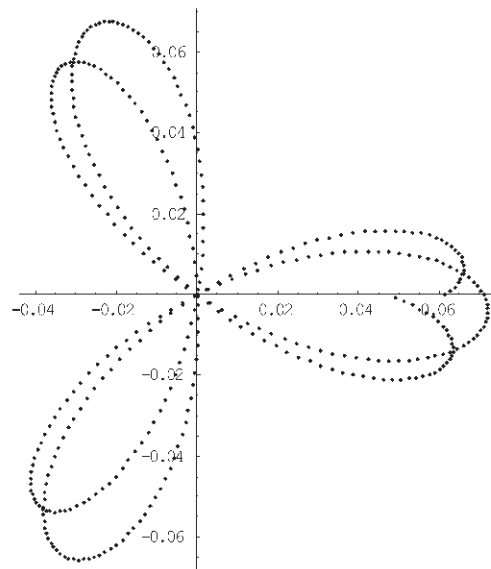
Figure 3. Polar plot of $y_1 (= x)$ for a typical case in which $0 \prec h_0 \prec 0.12$
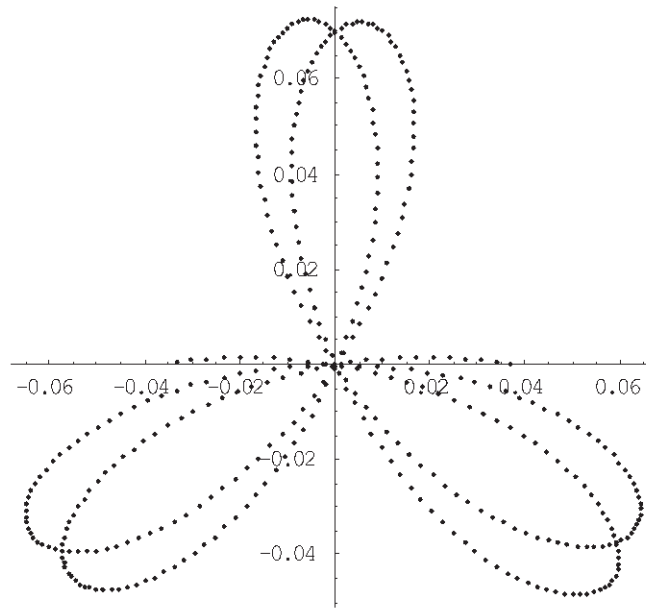
Figure 4. Polar plot of $y_2 (= \dot{x})$ for a typical case in which $0 \prec h_0 \prec 0.12$
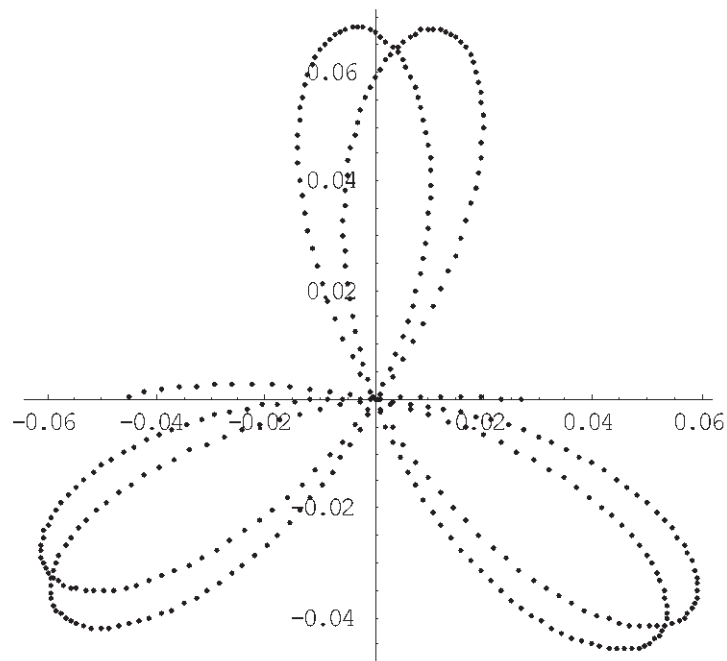
Figure 5. Polar plot of $y_3 (= y)$ for a typical case in which $0 \prec h_0 \prec 0.12$
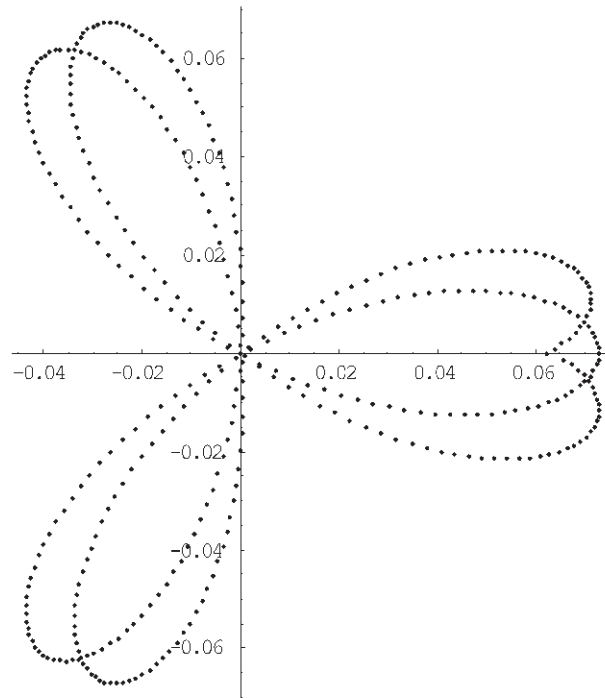
Figure 6: Polar plot of $y_4 (= \dot{y})$ for a typical case in which $0 \prec h_0 \prec 0.12$

CONCLUSIONS

The components of both the position and the velocity vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ are obtained as recurrent power series in time. Consequently, we can predict their values and hence their behavior at any time t^* (say) very simply and efficiently by using the continued fraction theory together with the device of interval division as mentioned above. On the other hand, the solution of the differential Equations (3) and (4) (or the system of Equation (6)) by numerical integration give us the components of both the position and the velocity vectors at definite values of t, according to the step size Δt, so if we need any of the components of $\mathbf{r}$ and $\dot{\mathbf{r}}$ at $t^* \neq t + \Delta t$ we must apply an interpolation formula. A process which needs more execution time. Moreover, which is the critical, the loss of accuracy that may arises due to the usage of the interpolation.

The analytical power series formulae for the determination of $\mathbf{r}$ and $\dot{\mathbf{r}}$ are *invariant* under many operations, because addition, multiplication, exponentiation, rising to power, differentiation, integration, etc. of a power series is also a power series. A fact which provides excellent flexibility in dealing with the analytical as well as the computational developments of problems related to the behavior of dynamical systems. On the other hand the numerical integration methods can not, by any way, provide such flexibility. Moreover, analytical solution of a problem usually offers much deeper insight into its nature.

The number of terms in the power series representations of the components of both the position and the velocity vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ can usually made small by means of the process of interval division as mentioned above.

Finally it was found that, the third integral of motion exists for only a limited range of initial conditions. But it may exist whatever the time interval may be, if it's initial value $\epsilon(0,0.12)$.

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Appendix A: Symbolic expressions of some of the A's,B's,C's and D's coefficients

$A_1 = a$

$A_2 = b$

$A_3 = \left(-\dfrac{1}{2} \; (a \; (2 \, c + 1)) \right)$

$A_4 = \left(\dfrac{1}{6} \; (-2 \, c \, b - b - 2 \, a \, d) \right)$

$A_5 = \left(\dfrac{1}{24} \; (2 \, a^3 + 2 \, c^2 \, a + 6 \, c \, a + a - 4 \, b \, d) \right)$

$A_6 = \left(\dfrac{1}{120} \; (b \; (10 \, a^2 - 2 \, c^2 + 10 \, c + 1) + 2 \, a \; (6 \, c + 5) \; d) \right)$

$A_7 =$
$\left(\dfrac{1}{720} \; (-4 \; (8 \, c + 5) \, a^3 + (16 \, c^3 - 20 \, c^2 - 22 \, c + 20 \, b^2 + 12 \, d^2 - 1) \, a + 4 \, b \; (2 \, c + 5) \; d \right.$

$A_8 = \dfrac{1}{5040} \; (20 \, b^3 -$
$(-24 \, c^3 + 8 \, c^2 + 42 \, c - 20 \, d^2 + 8 \, a^2 \; (23 \, c + 15) + 1) \, b - 14 \, a \; (4 \, a^2 - 4 \, c^2 + 8 \, c + 3)$

$A_9 = \dfrac{1}{40320} \; (56 \, a^5 + 2 \; (128 \, c^2 + 296 \, c + 81) \, a^3 -$
$392 \, b \, d \, a^2 + (8 \, c^4 - 176 \, c^3 + 162 \, c^2 + 72 \, d^2 \, c + 86 \, c - 132 \, d^2$
$- 4 \, b^2 \; (122 \, c + 75) + 1) \, a + 84 \, b \; (2 \, c^2 - 2 \, c - 1) \, d)$

$A_{10} = \dfrac{1}{362880}$
$(-4 \; (122 \, c + 75) \, b^3 - 1272 \, a \, d \, b^2 + (672 \, a^4 + 6 \; (360 \, c^2 + 752 \, c + 195) \, a^2 + 176 \, c^4$
$- 512 \, c^3 + 246 \, c^2 - 300 \, d^2 + 34 \, c \; (12 \, d^2 + 5) + 1) \, b +$
$2 \, a \, d \; (-80 \, c^3 - 384 \, c^2 + 462 \, c + 36 \, d^2 + 48 \, a^2 \; (12 \, c + 13) + 85))$

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Appendix A(Continued)

$$B_1 = b$$

$$B_2 = (-(a\,(1+2\,c)))$$

$$B_3 = \left(-\frac{b}{2} - b\,c - a\,d\right)$$

$$B_4 = \left(\frac{1}{6}\,(a + 2\,a^3 + 6\,a\,c + 2\,a\,c^2 - 4\,b\,d)\right)$$

$$B_5 = \left(\frac{1}{24}\,(b\,(1 + 10\,a^2 + 10\,c - 2\,c^2) + 2\,a\,(5 + 6\,c)\,d)\right)$$

$$B_6 = \left(\frac{1}{120}\right.$$
$$\left.(-4\,a^3\,(5 + 8\,c) + 4\,b\,(5 + 2\,c)\,d + a\,(-1 + 20\,b^2 - 22\,c - 20\,c^2 + 16\,c^3 + 12\,d^2))\right)$$

$$B_7 = \left(\frac{1}{720}\,(20\,b^3 - 14\,a\,(3 + 4\,a^2 + 8\,c - 4\,c^2)\,d - \right.$$
$$\left. b\,(1 + 42\,c + 8\,c^2 - 24\,c^3 + 8\,a^2\,(15 + 23\,c) - 20\,d^2))\right)$$

$$B_8 = \frac{1}{5040}\,(56\,a^5 + 2\,a^3\,(81 + 296\,c + 128\,c^2) - 392\,a^2\,b\,d + 84\,b\,(-1 - 2\,c + 2\,c^2)\,d + $$
$$a\,(1 + 86\,c + 162\,c^2 - 176\,c^3 + 8\,c^4 - 4\,b^2\,(75 + 122\,c) - 132\,d^2 + 72\,c\,d^2))$$

$$B_9 = \frac{1}{40320}\,(-4\,b^3\,(75 + 122\,c) - 1272\,a\,b^2\,d + $$
$$2\,a\,d\,(85 + 462\,c - 384\,c^2 - 80\,c^3 + 48\,a^2\,(13 + 12\,c) + 36\,d^2) + $$
$$b\,(1 + 672\,a^4 + 246\,c^2 - 512\,c^3 + 176\,c^4 + $$
$$6\,a^2\,(195 + 752\,c + 360\,c^2) - 300\,d^2 + 34\,c\,(5 + 12\,d^2)))$$

$$B_{10} = \frac{1}{362880}\,(-192\,a^5\,(10 + 13\,c) + 24\,a^2\,b\,(475 + 502\,c)\,d + $$
$$4\,b\,d\,(85 - 440\,b^2 + 504\,c - 930\,c^2 + 340\,c^3 + 120\,d^2) + $$
$$4\,a^3\,(-335 + 990\,b^2 - 2256\,c - 2580\,c^2 - 752\,c^3 + 234\,d^2) + $$
$$a\,(-1 + 1712\,c^3 + 240\,c^4 - 512\,c^5 + 24\,b^2\,(135 + 565\,c + 249\,c^2) + $$
$$1224\,d^2 - 20\,c^2\,(67 + 54\,d^2) - 6\,c\,(57 + 260\,d^2)))$$

Appendix A(Continued)

$$C_1 = c$$

$$C_2 = d$$

$$C_3 = \left(\frac{1}{2} \ (-a^2 + (-1 + c) \ c) \right)$$

$$C_4 = \left(\frac{1}{6} \ (-2 \ a \ b + (-1 + 2 \ c) \ d) \right)$$

$$C_5 = \left(\frac{1}{24} \ (-2 \ b^2 + c - 3 \ c^2 + 2 \ c^3 + a^2 \ (3 + 2 \ c) + 2 \ d^2) \right)$$

$$C_6 = \left(\frac{1}{120} \ (2 \ a \ b \ (5 + 6 \ c) - 2 \ a^2 \ d + (1 - 10 \ c + 10 \ c^2) \ d) \right)$$

$$C_7 = \left(\frac{1}{720} \ (2 \ a^4 - c + 11 \ c^2 - 20 \ c^3 + 10 \ c^4 + \right.$$
$$\left. 2 \ b^2 \ (5 + 6 \ c) - a^2 \ (11 + 20 \ c + 36 \ c^2) + 8 \ a \ b \ d - 10 \ d^2 + 20 \ c \ d^2) \right)$$

$$C_8 = \frac{1}{5040} \ (-14 \ a \ b \ (3 + 8 \ c + 8 \ c^2) - $$
$$8 \ a^2 \ (d + 16 \ c \ d) + d \ (-1 + 20 \ b^2 + 42 \ c - 120 \ c^2 + 80 \ c^3 + 20 \ d^2))$$

$$C_9 = \frac{1}{40320} \ (c - 43 \ c^2 + 162 \ c^3 - 200 \ c^4 + 80 \ c^5 + 8 \ a^4 \ (1 + 16 \ c) - $$
$$2 \ b^2 \ (21 + 66 \ c + 46 \ c^2) - 56 \ a \ b \ (3 + 10 \ c) \ d + 42 \ d^2 - 300 \ c \ d^2 + $$
$$300 \ c^2 \ d^2 + a^2 \ (43 - 20 \ b^2 + 162 \ c + 576 \ c^2 + 16 \ c^3 - 188 \ d^2))$$

$$C_{10} = \frac{1}{362880} \ (48 \ a^3 \ b \ (5 + 24 \ c) + 504 \ a^4 \ d + 6 \ a^2 \ (41 + 504 \ c + 32 \ c^2) \ d$$
$$- 2 \ a \ b \ (-85 + 20 \ b^2 - 462 \ c - 1128 \ c^2 + 80 \ c^3 + 468 \ d^2) + $$
$$d \ (1 - 170 \ c + 1170 \ c^2 - 2000 \ c^3 + 1000 \ c^4 - 12 \ b^2 \ (25 + 62 \ c) - 300 \ d^2 + 600 \ c \ d$$

Appendix A(Continued)

$$D_1 = d$$

$$D_2 = \left(-a^2 + (-1 + c)\, c\right)$$

$$D_3 = \left(-(a\, b) + \left(-\frac{1}{2} + c\right) d\right)$$

$$D_4 = \left(\frac{1}{6}\, (-2\, b^2 + c - 3\, c^2 + 2\, c^3 + a^2\, (3 + 2\, c) + 2\, d^2)\right)$$

$$D_5 = \left(\frac{1}{24}\, (2\, a\, b\, (5 + 6\, c) - 2\, a^2\, d + (1 - 10\, c + 10\, c^2)\, d)\right)$$

$$D_6 = \left(\frac{1}{120}\, (2\, a^4 - c + 11\, c^2 - 20\, c^3 + 10\, c^4 + 2\, b^2\, (5 + 6\, c) - \right.$$
$$\left. a^2\, (11 + 20\, c + 36\, c^2) + 8\, a\, b\, d - 10\, d^2 + 20\, c\, d^2)\right)$$

$$D_7 = \left(\frac{1}{720}\, (-14\, a\, b\, (3 + 8\, c + 8\, c^2) - 8\, a^2\, (d + 16\, c\, d) + \right.$$
$$\left. d\, (-1 + 20\, b^2 + 42\, c - 120\, c^2 + 80\, c^3 + 20\, d^2))\right)$$

$$D_8 = \frac{1}{5040}\, (c - 43\, c^2 + 162\, c^3 - 200\, c^4 + 80\, c^5 + 8\, a^4\, (1 + 16\, c) -$$
$$2\, b^2\, (21 + 66\, c + 46\, c^2) - 56\, a\, b\, (3 + 10\, c)\, d + 42\, d^2 - 300\, c\, d^2 +$$
$$300\, c^2\, d^2 + a^2\, (43 - 20\, b^2 + 162\, c + 576\, c^2 + 16\, c^3 - 188\, d^2))$$

$$D_9 = \frac{1}{40320}\, (48\, a^3\, b\, (5 + 24\, c) + 504\, a^4\, d + 6\, a^2\, (41 + 504\, c + 32\, c^2)\, d -$$
$$2\, a\, b\, (-85 + 20\, b^2 - 462\, c - 1128\, c^2 + 80\, c^3 + 468\, d^2) +$$
$$d$$
$$(1 - 170\, c + 1170\, c^2 - 2000\, c^3 + 1000\, c^4 - 12\, b^2\, (25 + 62\, c) - 300\, d^2 + 600\, c\, d^2))$$

$$D_{10} = \frac{1}{362880}\, (-504\, a^6 - 40\, b^4 - c + 171\, c^2 - 1340\, c^3 + 3170\, c^4 - 3000\, c^5 + 1000\, c^6 -$$
$$6\, a^4\, (81 + 860\, c + 332\, c^2) + 5040\, a^3\, b\, d + 1008\, a\, b\, (2 + 15\, c + c^2)\, d$$
$$- 170\, d^2 + 3240\, c\, d^2 - 8700\, c^2\, d^2 + 5800\, c^3\, d^2 + 600\, d^4 -$$
$$2\, b^2\, (-85 - 612\, c - 1350\, c^2 + 452\, c^3 + 840\, d^2) +$$
$$a^2\, (-171 - 1340\, c - 8052\, c^2 +$$
$$480\, c^3 - 488\, c^4 + 60\, b^2\, (19 + 74\, c) + 4860\, d^2 + 456\, c\, d^2))$$

اتجاه رمزي لمدى صلاحية التكامل الثالث للحركة

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الملخص

تم في هذه البحث إيجاد حلول حرفية رمزية في صور متسلسلات قوى تعاودية للحركة المستوية في مجال متماثل محورياً وذلك لدراسة مدى صلاحية التكامل الثالث للحركة. ولحساب هذه الحلول استخدمت طريقة فعالة طوعت فيها نظرية الكسر المستمر مع أداة تقسيم الفترة. احتوى البحث أيضاً على بعض النتائج العددية والأشكال التوضيحية للحلول، وقد توصلنا إلى النتيجة بوجود التكامل الثالث للحركة وذلك لمدى محدود للشروط الأولية ويمكن وجوده أيضا في أي فترة زمنية طالما أن له قيمة محصورة في المدى (0, 0.12).