Natural cubic spline for parabolic equation with constant and variable coefficients

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Abstract: This research tackles solving a broad class of second-order partial differential equations (PDEs) with a novel method using natural cubic splines. These equations, crucial in science and engineering, describe phenomena like heat flow and wave propagation. The method works for both parabolic (diffusion) and hyperbolic (wave) equations, with a focus here on parabolic ones. The key idea lies in approximating the spatial derivatives in the PDE with the second derivative of a natural cubic spline function. Imagine a smooth curve broken into segments; natural cubic splines ensure these segments connect seamlessly while having zero second derivative at the joints. This makes them ideal for mimicking the solution's behavior. For the time derivatives, the paper employs a finite difference method. This approximates the derivative based on the function's values at specific time steps. By combining these approximations, the original PDE transforms into a system of solvable algebraic equations. The paper explores solving this system explicitly (directly calculating new solutions based on previous ones) and implicitly (solving a system of equations at each step). This offers flexibility, with explicit schemes being faster but potentially less stable, while implicit schemes provide more stability but require more computation. Finally, the paper validates the method's effectiveness through numerical examples with various boundary conditions (specifying the solution's behavior at domain edges). This showcases the method's applicability in real-world scenarios with different constraints. In conclusion, this research offers a valuable tool for solving diverse second-order PDEs. The method's ability to handle both constant and variable coefficients and its exploration of different solution strategies make it a versatile and adaptable approach.

Keywords: Second-order Parabolic equation; Natural Cubic Spline; Finite difference scheme; Absolute errors.

1. Introduction:

Boundary value problems are encountered in many areas of science, engineering and technology. Therefore, finding solution of these BVP's is a fundamental challenge to the researchers. Hence choosing the numerical method to find more accurate solution plays an important role in their physical significance. With this reason, we developed natural cubic spline method to solve various boundary value problems of differential equations with different Dirichlet, Neumann and Robin conditions. NCS procedure is developed to solve different linear, nonlinear ordinary and partial differential equations.

Partial differential equations (PDE) are very important in many branches of mathematics, science and engineering etc such as elasticity, hydrodynamics, quantum mechanics electromagnetic theory and etc. They also arise in diverse fields such as biology, physics, differential geometry, control theory, metrology, material science, electro-magnetic theory, aeronautics, nuclear physics, medicine, electro-dynamics, elasticity, fluid dynamics, diffusion of chemicals, vibrations of solids, spread of heat, interactions of photons, structure of molecules, flow of fluids, interactions of electrons, and radiation of electromagnetic wave describe PDEs. Its uses spread into economics, financial forecasting, image processing, flows in porous media, turbulent transport problems and many other fields.

One dimension infinite solid, temperature distribution in a rod and in the bar of uniform cross section, transverse vibration of a uniform flexible beam are some examples of Parabolic PDE. parabolic PDE has many applications in chemical separation processes, computational hydraulics, ground water pollution problems, problems related to spread of contaminants in fluids and etc. as heat transfer in draining films [1], dispersion of pollutants in rivers and streams [2], thermal pollution in river systems [3], flow in porous media, spread of contaminants in coastal seas and estuaries, etc are some applications of parabolic PDE.

We considered general second order PDE of the form:

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + E\frac{\partial u}{\partial x} + Fu + G(x,t), \quad 0 \le x \le 1,\ t > 0 \tag{1.1}$$

With the initial conditions

$$\begin{cases} u(x,0) = f(x), & x \in [a,b] \\ u_t(x,0) = g(x), & x \in [a,b] \end{cases}$$

And the boundary conditions

$$\begin{cases} u(a,t) = l(x), & x \in [a,b] \\ u(b,0) = m(x), & x \in [a,b] \end{cases}$$

The time derivatives in (1.1) are replaced by a central finite difference operator and the space derivatives are replaced by natural cubic spline at the point (ih, jk).

Numerous numerical techniques have recently been developed to solve parabolic PDEs. M.M Butt [5] solved heat equation with variable coefficients. Suayip Yuzbasi [6] devised a new collocation approach based on Bessel functions of the first kind for the solution of linear 2nd-order PDEs with variable coefficients under various boundary conditions. The adaptive grid Haar wavelet collocation approach was used by Shiralashetty [7] to solve parabolic partial differential equations numerically.

We considered NCS Explicit and Implicit method in solving equation (1.1). Parabolic PDE's with different types of boundary conditions are considered and obtained a tri-diagonal system of (n+1) equations in (n+1) unknowns and represented in matrix form. It is explained in detail how the tri diagonal matrix form will change with given boundary conditions. Exact solutions are contrasted with examples of parabolic PDE utilising the explicit and implicit NCS methods. Table values for various step sizes are provided to evaluate the precision of the suggested methodology.

2. Natural Cubic Spline:

Let the cubic spline $S(x)$ interpolates $y(x)$ at the mesh $a = x_0 < x_1 < ...... < x_n = b$.

Since $S(x)$ is piecewise cubic spline, its second order derivative $S''(x)$ is piecewise linear on the interval $[x_{i-1}, x_i]$.

Using linear Lagrange interpolating formula we have

$$S''(x) = S''(x_{i-1})\frac{x_i - x}{x_i - x_{i-1}} + S''(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}}$$

Putting $M_i = S''(x_i)$ and $M_{i-1} = S''(x_{i-1})$, the above expression becomes

$$S''(x) = \frac{1}{h}\left(M_{i-1}(x_i - x) + M_i(x - x_{i-1})\right) \tag{2.1}$$

Integrating (2.1) twice, we get

$$S(x) = M_{i-1}\frac{(x_i - x)^3}{6h} + M_i\frac{(x - x_{i-1})^3}{6h} + C_1 x + C_2 \tag{2.2}$$

Where C_1 and C_2 are constants of integration to be determined

Evaluating $S(x)$ at x_i and x_{i-1} we have

$$y_{i-1} = M_{i-1}\frac{h^2}{6} + C_1 x_{i-1} + C_2 \tag{2.3}$$

$$y_i = M_i\frac{h^2}{6} + C_1 x_i + C_2 \tag{2.4}$$

Solving (2.3) and (2.4) for C_1 and C_2 we get

$$C_1 = (\frac{h}{6}M_{i-1} - \frac{h}{6}M_i) + \frac{(y_i - y_{i-1})}{h}$$

$$C_2 = y_i - \frac{h^2}{6}M_i - \left[\frac{h}{6}(M_{i-1} - M_i) + \frac{y_i - y_{i-1}}{h}\right]x_i$$

Substituting the values of C_1 and C_2 in (2.4) we have

$$\begin{aligned} S(x) = {} & \frac{1}{6h}\left(M_{i-1}\left(x_i - x\right)^3 + M_i\left(x - x_{i-1}\right)^3\right) + \left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right)\left(\frac{x_i - x}{h}\right) \\ & + \left(y_i - \frac{h^2}{6}M_i\right)\left(\frac{x - x_{i-1}}{h}\right) \end{aligned} \tag{2.5}$$

The function $S(x)$ in the interval $[x_i, x_{i+1}]$ is obtained by replacing i by $i+1$ in equation (2.5)

Hence

$$\begin{aligned} S(x) = {} & M_i\frac{\left(x_{i+1} - x\right)^3}{6h} + M_i\frac{\left(x - x_i\right)^3}{6h} + \left(y_i - \frac{h^2}{6}M_i\right)\left(\frac{x_{i+1} - x}{h}\right) \\ & + \left(y_{i+1} - \frac{h^2}{6}M_{i+1}\right)\left(\frac{x - x_i}{h}\right) \end{aligned} \tag{2.6}$$

Differentiating (2.5) and (2.6)

$$S'(x) = \frac{1}{2h}\left(-M_{i-1}(x_i - x)^2 + M_i(x - x_{i-1})^2\right) + \frac{y_i - y_{i-1}}{h} - \frac{(M_i - M_{i-1})}{6}h \tag{2.7}$$

$$S'(x) = -M_i\frac{(x_{i+1} - x)^2}{2h} + M_{i+1}\frac{(x - x_i)^2}{2h} + \frac{y_{i+1} - y_i}{h} - \frac{(M_i + M_{i+1})}{6}h \tag{2.8}$$

calculating $S'(x)$ at $x = x_i$

$$S'(x_i^-) = \frac{h}{6}M_{i-1} + \frac{h}{3}M_i + \frac{y_i - y_{i-1}}{h}, \quad i = 1, 2, \ldots .n. \tag{2.9}$$

$$S'(x_i^+) = -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{y_{i+1} - y_i}{h}, \quad i = 0, 1, \ldots, n-1. \tag{2.10}$$

Using continuity condition of the cubic spline, we have

$$\frac{h^2}{6}\left(M_{i-1}+4M_i+M_{i+1}\right)=(y_{i-1}-2y_i+y_{i+1}),\ i=1,2,...,n. \qquad (2.11)$$

The above relation is called the continuity or consistency relations of the cubic spline.

The cubic spline can be assumed as

$$S_j(x)=M_{i-1}^j\frac{(x_i-x)^3}{6h}+M_i^j\frac{(x-x_{i-1})^3}{6h}+\left(u_{i-1}^j-\frac{h^2}{6}M_{i-1}^j\right)\left(\frac{x_i-x}{h}\right)$$
$$+\left(u_i^j-\frac{h^2}{6}M_i^j\right)\left(\frac{x-x_{i-1}}{h}\right),\ i=1,2,..,n.$$

Let $L_i^j=S'\left(x_i^+\right)=-\frac{h}{3}M_i^j-\frac{h}{6}M_{i+1}^j+\frac{u_{i+1}^j-u_i^j}{h},\quad i=0,1,\cdots,n-1 \qquad (2.12)$

$$L_i^j=S'\left(x_i^-\right)=\frac{h}{3}M_i^j+\frac{h}{6}M_{i-1}^j+\frac{u_i^j-u_{i-1}^j}{h},\quad i=0,1,\ldots,n. \qquad (2.13)$$

From (2.12) and (2.13) we have

$$-L_i^j-\frac{h}{6}M_{i+1}^j+\frac{u_{i+1}^j-u_i^j}{h}=\frac{h}{3}M_i^j,\quad i=0,1,\cdots,n-1 \qquad (2.14)$$

$$L_i^j-\frac{h}{6}M_{i-1}^j-\frac{u_i^j-u_{i-1}^j}{h}=\frac{h}{3}M_i^j,\quad i=0,1,\ldots,n. \qquad (2.15)$$

Equating (2.14) and (2.15)

$$-L_i^j-\frac{h}{6}M_{i+1}^j+\frac{u_{i+1}^j-u_i^j}{h}=L_i^j-\frac{h}{6}M_{i-1}^j-\left(\frac{u_i^j-u_{i-1}^j}{h}\right)$$

$$\Rightarrow -L_i^j-\frac{1}{2}\left[L_{i+1}^j-\frac{h}{6}M_i^j-\left(\frac{u_{i+1}^j-u_i^j}{h}\right)\right]+\left(\frac{u_{i+1}^j-u_i^j}{h}\right)=L_i^j+\frac{1}{2}\left[L_{i-1}^j+\frac{h}{6}M_i^j-\left(\frac{u_i^j-u_{i-1}^j}{h}\right)\right]$$
$$-\left(\frac{u_i^j-u_{i-1}^j}{h}\right)$$

$$\Rightarrow 2l_i^j+\frac{1}{2}L_{i-1}^j+\frac{1}{2}L_{i+1}^j-\frac{1}{2}\left(\frac{u_i^j-u_{i-1}^j}{h}\right)-\frac{1}{2}\left(\frac{u_{i+1}^j-u_i^j}{h}\right)-\left(\frac{u_i^j-u_{i-1}^j}{h}\right)-\left(\frac{u_{i+1}^j-u_i^j}{h}\right)=0$$

$$\Rightarrow 2l_i^j+\frac{1}{2}L_{i-1}^j+\frac{1}{2}L_{i+1}^j-\frac{1}{h}\left[\frac{u_i^j}{2}-\frac{u_{i-1}^j}{2}+u_i^j-u_{i-1}^j\right]-\frac{1}{h}\left[\frac{u_{i+1}^j}{2}-\frac{u_i^j}{2}+u_{i+1}^j-u_i^j\right]=0$$

$$\Rightarrow 2l_i^j+\frac{1}{2}L_{i-1}^j+\frac{1}{2}L_{i+1}^j-\frac{1}{h}\left[\frac{3u_i^j}{2}-\frac{3u_{i-1}^j}{2}\right]-\frac{1}{h}\left[\frac{3u_{i+1}^j}{2}-\frac{3u_i^j}{2}\right]=0$$

$$\Rightarrow \; 4L_i^j + L_{i-1}^j + L_{i+1}^j - \frac{1}{h}\left[3u_i^j - 3u_{i-1}^j + 3u_{i+1}^j - 3u_i^j\right] = 0$$

$$\Rightarrow \; L_{i-1}^i + L_{i+1}^j + 4L_i^j = \frac{1}{h}\left[3u_{i+1}^j - 3u_{i-1}^j\right]$$

Dividing by 6 through out

$$\frac{1}{6}L_{i-1}^j + \frac{2}{3}L_i^j + \frac{1}{6}L_{i+1}^j = \frac{1}{2h}\left(u_{i+1}^j - u_{i-1}^j\right)$$

This is called recurrence relation in L_i^j

3. NCS PROCEDURE FOR PDE

Consider second-order PDE of the form:

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + E\frac{\partial u}{\partial x} + Fu + G(x,t), \quad 0 \le x \le 1, t > 0 \tag{3.1}$$

The time derivatives in (3.1) are replaced by a central finite difference operator and the space derivatives are replaced by natural cubic spline at the point (ih, jk)

$$A\left(\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2}\right) + B\left(\frac{u_i^{j+1} - u_i^{j-1}}{2k}\right) = DM_i^j + EL_i^j + Fu_i^j + G_i^j,$$
$$i = 0,1,...n, \quad j = 1,2,,..., \; nh = 1$$

where $L_i^j = S'_j(x_i), \; M_i^j = S''_j(x_i)$

Let $M_i^i = \left(\dfrac{M_i^{j-1} + M_i^{j+1}}{2}\right)$ and $L_i^i = \left(\dfrac{L_i^{j-1} + L_i^{j+1}}{2}\right)$ then above equation becomes

$$A\left(\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2}\right) + B\left(\frac{u_i^{j+1} - u_i^{j-1}}{2k}\right) = D\left(\frac{M_i^{j-1} + M_i^{j+1}}{2}\right) + E\left(\frac{L_i^{j-1} + L_i^{j+1}}{2}\right) + Fu_i^j + G_i^j, \tag{3.2}$$

From (2.11), recurrence relation in M_i^j

$$\frac{1}{6}M_{i-1}^j + \frac{2}{3}M_i^j + \frac{1}{6}M_{i+1}^j = \left(\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2}\right), \quad i = 1,2,\dots,n-1. \tag{3.3}$$

Similarly, recurrence relation in L_i^j

$$\frac{1}{6}L_{i-1}^{j}+\frac{2}{3}L_i^{j}+\frac{1}{6}L_{i+1}^{j}=\left(\frac{u_{i+1}^{j}-u_{i-1}^{j}}{2h}\right),\ \ i=1,2,\ldots,n-1. \tag{3.4}$$

Similarly, for $(j-1)$th and $(j+1)$th time levels from (3.3), we have

$$\frac{1}{6}M_{i-1}^{j-1}+\frac{2}{3}M_i^{j-1}+\frac{1}{6}M_{i+1}^{j-1}=\left(\frac{u_{i-1}^{j-1}-2u_i^{j-1}+u_{i+1}^{j-1}}{h^2}\right),\ \ i=1,2,\ldots,n-1. \tag{3.5}$$

$$\frac{1}{6}M_{i-1}^{j+1}+\frac{2}{3}M_i^{j+1}+\frac{1}{6}M_{i+1}^{j+1}=\left(\frac{u_{i-1}^{j+1}-2u_i^{j+1}+u_{i+1}^{j+1}}{h^2}\right),\ \ i=1,2,\ldots,n-1. \tag{3.6}$$

Similarly, for $(j-1)$th and $(j+1)$th time levels from (3.4), we have

$$\frac{1}{6}L_{i-1}^{j-1}+\frac{2}{3}L_i^{j-1}+\frac{1}{6}L_{i+1}^{j-1}=\left(\frac{u_{i+1}^{j-1}-u_{i-1}^{j-1}}{2h}\right),\ \ i=1,2,\ldots,n-1. \tag{3.7}$$

$$\frac{1}{6}L_{i-1}^{j+1}+\frac{2}{3}L_i^{j+1}+\frac{1}{6}L_{i+1}^{j+1}=\left(\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2h}\right),\ \ i=1,2,\ldots,n-1. \tag{3.8}$$

Multiplying (3.5) and (3.6) by D/2 and again (3.7) and (3.8) by E/2 and substitute in (3.2), we obtain

$$\begin{aligned}
&\frac{D}{2}\left(\frac{1}{6}M_{i-1}^{j-1}+\frac{2}{3}M_i^{j-1}+\frac{1}{6}M_{i+1}^{j-1}=\left(\frac{u_{i-1}^{j-1}-2u_i^{j-1}+u_{i+1}^{j-1}}{h^2}\right)\right).+\\
&\frac{D}{2}\left(\frac{1}{6}M_{i-1}^{j+1}+\frac{2}{3}M_i^{j+1}+\frac{1}{6}M_{i+1}^{j+1}=\left(\frac{u_{i-1}^{j+1}-2u_i^{j+1}+u_{i+1}^{j+1}}{h^2}\right)\right)+\\
&\frac{E}{2}\left(\frac{1}{6}L_{i-1}^{j-1}+\frac{2}{3}L_i^{j-1}+\frac{1}{6}L_{i+1}^{j-1}=\left(\frac{u_{i+1}^{j-1}-u_{i-1}^{j-1}}{2h}\right)\right)+\\
&\frac{E}{2}\left(\frac{1}{6}L_{i-1}^{j+1}+\frac{2}{3}L_i^{j+1}+\frac{1}{6}L_{i+1}^{j+1}=\left(\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2h}\right)\right).
\end{aligned} \tag{3.9}$$

Simplifying,

$$
\begin{aligned}
&\frac{1}{6}\left(D\left(\frac{M_{i-1}^{j-1}+M_{i-1}^{j+1}}{2}\right)+E\left(\frac{L_{i-1}^{j-1}+L_{i-1}^{j+1}}{2}\right)\right)\\
&+\frac{2}{3}\left(D\left(\frac{M_{i}^{j-1}+M_{i}^{j+1}}{2}\right)+E\left(\frac{L_{i}^{j-1}+L_{i}^{j+1}}{2}\right)\right)\\
&+\frac{1}{6}\left(D\left(\frac{M_{i+1}^{j-1}+M_{i+1}^{j+1}}{2}\right)+E\left(\frac{L_{i+1}^{j-1}+L_{i+1}^{j+1}}{2}\right)\right)=\\
&\frac{D}{2}\left(\frac{u_{i-1}^{j-1}-2u_{i}^{j-1}+u_{i+1}^{j-1}}{h^2}\right)+\frac{D}{2}\left(\frac{u_{i-1}^{j+1}-2u_{i}^{j+1}+u_{i+1}^{j+1}}{h^2}\right)\\
&+\frac{E}{2}.\left(\frac{u_{i+1}^{j-1}-u_{i-1}^{j-1}}{2h}\right)+\frac{E}{2}\left(\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2h}\right)
\end{aligned}
\tag{3.10}
$$

Eliminating M_i^j and L_i^j from (3.2) and (3.10), we get

$$
\begin{aligned}
&\left(A\left(u_{i-1}^{j+1}-2u_{i-1}^{j}+u_{i-1}^{j-1}\right)+\frac{Bk}{2}\left(u_{i-1}^{j+1}-u_{i-1}^{j-1}\right)+Ck^2u_{i-1}^{j}-Fk^2u_{i-1}^{j}-k^2G_{i-1}^{j}\right)+\\
&4\left(A\left(u_{i}^{j+1}-2u_{i}^{j}+u_{i}^{j-1}\right)+\frac{Bk}{2}\left(u_{i}^{j+1}-u_{i}^{j-1}\right)+Ck^2u_{i}^{j}-Fk^2u_{i}^{j}-k^2G_{i}^{j}\right)+\\
&\left(A\left(u_{i+1}^{j+1}-2u_{i+1}^{j}+u_{i+1}^{j-1}\right)+\frac{Bk}{2}\left(u_{i+1}^{j+1}-u_{+1i}^{j-1}\right)+Ck^2u_{i+1}^{j}-Fk^2u_{+1}^{j}-k^2G_{i+1}^{j}\right)=\\
&\frac{3Dk^2}{h^2}\left(u_{i-1}^{j-1}-2u_{i}^{j-1}+u_{i+1}^{j-1}\right)+\frac{3Dk^2}{h^2}\left(u_{i-1}^{j+1}-2u_{i}^{j+1}+u_{i+1}^{j+1}\right)\\
&\qquad+\frac{3Ek^2}{2h}.\left(u_{i+1}^{j-1}-u_{i-1}^{j-1}\right)+\frac{3Ek^2}{2h}\left(u_{i+1}^{j+1}-u_{i-1}^{j+1}\right)
\end{aligned}
\tag{3.11}
$$

Finally, a tri-diagonal system of equations,

$$
\begin{aligned}
&\left(A+\frac{BK}{2}-\frac{3Dk^2}{h^2}+\frac{3Ek^2}{2h}\right)u_{i-1}^{j+1}+4\left(A+\frac{BK}{2}+\frac{3Dk^2}{2h^2}\right)u_{i}^{j+1}\\
&+\left(A+\frac{BK}{2}-\frac{3Dk^2}{h^2}-\frac{3Ek^2}{2h}\right)u_{i+1}^{j+1}=\left(2A+Fk^2\right)u_{i-1}^{j}+4\left(2A+Fk^2\right)u_{i}^{j}\\
&+\left(2A+Fk^2\right)u_{i+1}^{j}-\left(A-\frac{BK}{2}-\frac{3Dk^2}{h^2}+\frac{3Ek^2}{2h}\right)u_{i-1}^{j-1}-4\left(A-\frac{BK}{2}+\frac{3Dk^2}{2h^2}\right)u_{i}^{j-1}\\
&-\left(A-\frac{BK}{2}-\frac{3Dk^2}{h^2}-\frac{3Ek^2}{2h}\right)u_{i+1}^{j-1}+(k^2G_{i-1}^{j}+k^2G_{i}^{j}+k^2G_{i+1}^{j})
\end{aligned}
\tag{3.12}
$$

On simplification,

$$
\begin{aligned}
&\left(l-3Dr^2+\frac{3Erk}{2}\right)u_{i-1}^{j+1}+4\left(l+\frac{3Dr^2}{2}\right)u_i^{j+1}+\left(l-3Dr^2-\frac{3Erk}{2}\right)u_{i+1}^{j+1}=\\
&\left(m+Fk^2\right)u_{i-1}^{j}+4\left(m+Fk^2\right)u_i^{j}+\left(m+Fk^2\right)u_{i+1}^{j}\\
&-\left(n-3Dr^2+\frac{3Erk}{2}\right)u_{i-1}^{j-1}-4\left(n+\frac{3Dr^2}{2}\right)u_i^{j-1}-\left(n-3Dr^2-\frac{3Erk}{2}\right)u_{i+1}^{j-1}\\
&\qquad\qquad+\left(k^2G_{i-1}^{j}+k^2G_i^{j}+k^2G_{i+1}^{j}\right)
\end{aligned}
\tag{3.13}
$$

where $l=A+\dfrac{Bk}{2},\quad m=2A,\quad n=A-\dfrac{BK}{2},\quad r=\dfrac{k}{h}.$

4. NUMERICAL RESULTS

In this section we have considered parabolic PDE with different types of boundary conditions

EXAMPLE 1: Consider the heat equation of the form:

$$u_t=u_{xx},\quad 0<x<L,\ t>0 \tag{4.1}$$

subject to:

$u(x,0)=100\sin(\pi x/80),\ 0\le x\le L; L=80cm,\ u(0,t)=u(1,t)=0,\ t\ge 0$

Exact solution: $u(x,t)=100\sin(\pi x/80)e^{-c^2\pi^2 t/L^2}$

NATURAL CUBIC SPLINE EXPLICIT METHOD

Let x denote the space variable ant t denote the time variable and $c=1$

Replacing time derivative by forward difference operator and space derivatives by natural cubic spline in equation (4.1) explicitly, we get

$$\frac{u_i^{j+1}-u_i^j}{k}=M_i^j \tag{4.2}$$

From (2.11),

$$\frac{1}{6}M_{i-1}^j+\frac{2}{3}M_i^j+\frac{1}{6}M_{i+1}^j=\left(\frac{u_{i-1}^j-2u_i^j+u_{i+1}^j}{h^2}\right),\ \ i=1,2,\ldots,n-1. \tag{4.3}$$

Using (4.3), equation (4.2) becomes

$$u_{i-1}^{j+1}+4u_i^{j+1}+u_{i+1}^{j+1}=(1+6r)u_{i-1}^j+4(1-3r)u_i^j+(1+6r)u_{i+1}^j,\ \ i=1,2,\ldots,n-1. \tag{4.4}$$

Equation (4.4) is known as **NATURAL CUBIC SPLINE EXPLICIT** recurrence relation to solve equation (4.1). Using known values at j^{th} level, unknown values can be obtained at $(j+1)^{th}$ level. For different index values i and j, a tri-diagonal system of n+1 equation in n+1 unknown is obtained from eqn. (4.4) and represents in matrix form as

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & . & . & . & . & . & . & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & l & m \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j} \\ u_{N_x}^{j} \end{bmatrix} \tag{4.5}$$

for $i = 0,1,2,....,N_x$ and $j = 0,1,2,...,N_t$,

where $l = 1+6r$, $m = 4(1-3r)$ and $n = 1+6r$.

Since the conditions, $u(0,t) = 0 \Rightarrow u_0^j = 0,\ j \ge 0$, first equation of (4.4) becomes

$$u_0^{j+1} + 4u_1^{j+1} + u_2^{j+1} = (1+6r)u_0^j + 4(1-3r)u_1^j + (1+6r)u_2^j,\ \ i = 1$$
$$\Rightarrow 4u_1^{j+1} + u_2^{j+1} = 4(1-3r)u_1^j + (1+6r)u_2^j,$$

and $u(L,t) = 0 \Rightarrow u_{N_x}^j = 0,\ j \ge 0$, last equation in (4.4) becomes

$$u_{n-2}^{j+1} + 4u_{n-1}^{j+1} + u_n^{j+1} = (1+6r)u_{n-2}^j + 4(1-3r)u_{n-1}^j + (1+6r)u_n^j,\ \ i = n-1.$$
$$\Rightarrow u_{n-2}^{j+1} + 4u_{n-1}^{j+1} = (1+6r)u_{n-2}^j + 4(1-3r)u_{n-1}^j$$

Hence the above matrix (4.5) reduces to

$$\begin{bmatrix} 4 & 1 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & . & . & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & . & . & 0 & 0 \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ 0 & 0 & 0 & . & . & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & . & . & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-2}^{j+1} \\ u_{N_x-1}^{j+1} \end{bmatrix} = \begin{bmatrix} m & n & 0 & . & . & . & 0 & 0 & 0 \\ l & m & n & 0 & . & . & 0 & 0 & 0 \\ 0 & l & m & n & 0 & . & . & 0 & 0 \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ 0 & 0 & 0 & . & . & 0 & l & m & n \\ 0 & 0 & 0 & . & . & 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} u_1^{j} \\ u_2^{j} \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-2}^{j} \\ u_{N_x-1}^{j} \end{bmatrix}$$

In short, $M_L X^{j+1} = M_R X^j$

Hence, the required solution is given by

$$X^{j+1} = \left(M_L\right)^{-1}\left[M_R X^j\right] \tag{4.6}$$

By the inverse operation, solution is obtained and presented in fig. 4.1. The NCS solution is coinciding with analytical solution and presented in fig.4.1. It shows that NCS method results are correlated with analytical solution. To check the accuracy of the NCS method, absolute error at t = 0.5 is calculated and presented in table 4.1 at different step sizes along space coordinates. It is noticed that at step size 10^{-3}, accuracy of 10^{-8} is obtained for NCS method. It is also observed from table 4.1 that if $r < 1$, NCS is giving stable solutions. If $r > 1$step size along t is 0.01 and step size along x is 0.001, convergence of the results is not obtained. Further NCS method is also compared with finite difference method (FDM). It is clear that difference between NCS method results and FDM is negligible and conclude that difficult to judge which is far better. Therefore, we extend the investigation for **NCS IMPLICIT** method.

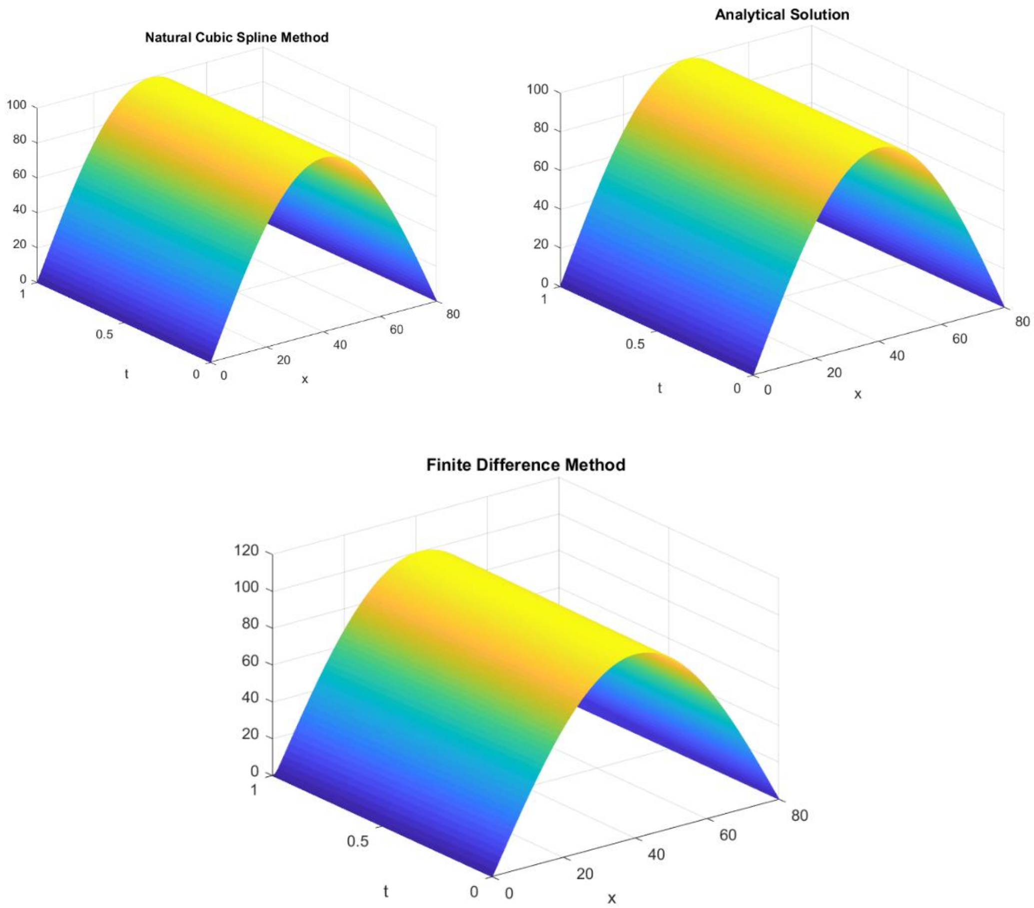

Figure 4.1: Solution of example 4.1 using NCS explicit method and analytical solution

Table 4.1: Absolute Error at t = 0.5 with 0.1 step size along t

Step size along x	r	Absolute Error	
		Finite Difference Method	NCS method
1/10	0.00015	6.3635e-04	6.3102e-04
1/50	0.003906	2.5945e-05	2.4750e-05
1/100	0.015625	6.9312e-06	5.7427e-06

1/1000	1.5625	1.2239e+48 (Unstable)	6.7244e+21 (Unstable)

NATURAL CUBIC SPLINE WITH IMPLICIT METHOD

In implicit method, replacing time derivative by forward difference operator and space derivatives by the average of $M_i^j = S_j''(x_i)$ based on natural cubic spline in equation (4.1) we have

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{1}{2}(M_i^j + M_i^{j+1}) \tag{4.7}$$

From (2.2),

$$\frac{1}{6}M_{i-1}^j + \frac{2}{3}M_i^j + \frac{1}{6}M_{i+1}^j = \left(\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2}\right), \quad i = 1, 2, \dots, n-1. \tag{4.8}$$

At $(j+1)th$ level we have

$$\frac{1}{6}M_{i-1}^{j+1} + \frac{2}{3}M_i^{j+1} + \frac{1}{6}M_{i+1}^{j+1} = \left(\frac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2}\right), \quad i = 1, 2, \dots, n-1. \tag{4.9}$$

Using (4.8) and (4.9) equation (4.7) becomes

$$\begin{aligned}(1-3r)u_{i-1}^{j+1} + (4+6r)u_i^j + (1-3r)u_{i+1}^{j+1} \\ = (1+3r)u_{i-1}^j + (4-6r)u_i^j + (1+3r)u_{i+1}^j, \quad r = \frac{k}{h^2}.\end{aligned} \tag{4.10}$$

Equation (4.10) is known as "**NATURAL CUBIC SPLINE IMPLICIT"** recurrence relation to solve equation (4.1). Using known values at j^{th} level, unknown values can be obtained at $(j+1)^{th}$ level. For different indices i and j, a tri-diagonal system of n+1 equation in n+1 unknown is obtained from eqn. (4.10) and represents in matrix form as

$$\begin{bmatrix} a & b & c & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & a & b & c & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & a & b & c & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & a & b & c \\ 0 & 0 & . & . & . & . & . & . & 0 & a & b \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \cdot \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & l & m \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ \cdot \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j} \\ u_{N_x}^{j} \end{bmatrix}$$

for $i = 0,1,2,....,N_x$, and $j = 0,1,2,...,N_t$.

where $a = 1-3r, b = 4+6r$ and $c = 1-3r$; $l = 1+3r$, $m = 4-6r$ and $n = 1+3r$.

Since $u(0,t) = 0 \Rightarrow u_0^j = 0,\ j \geq 0$, first equation in (4.10) becomes

$$(1-3r)u_0^{j+1} + (4+6r)u_1^{j+1} + (1-3r)u_2^{j+1} = (1+3r)u_0^j + (4-6r)u_1^j + (1+3r)u_2^j,\ \ i = 1$$
$$\Rightarrow (4+6r)u_1^{j+1} + (1-3r)u_2^{j+1} = (4-6r)u_1^j + (1+3r)u_2^j,$$

and $u(L,t) = 0 \Rightarrow u_{N_x}^j = 0,\ j \geq 0$, last equation in (4.10) becomes

$$(1-3r)u_{n-2}^{j+1} + (4+6r)u_{n-1}^{j+1} + (1-3r)u_n^{j+1} = (1+3r)u_{n-2}^j + (4-6r)u_{n-1}^j + (1+3r)u_n^j,\ \ i = n-1$$
$$\Rightarrow (1-3r)u_{n-2}^{j+1} + (4+6r)u_{n-1}^{j+1} = (1+3r)u_{n-2}^j + (4-6r)u_{n-1}^j,$$

Hence the above matrix reduces to

$$\begin{bmatrix} b & c & 0 & . & . & . & 0 & 0 & 0 \\ a & b & c & 0 & . & . & 0 & 0 & 0 \\ 0 & a & b & c & 0 & . & . & 0 & 0 \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ 0 & 0 & 0 & . & . & 0 & a & b & c \\ 0 & 0 & 0 & . & . & 0 & 0 & a & b \end{bmatrix} \begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \cdot \\ . \\ . \\ . \\ . \\ u_{N_x-2}^{j+1} \\ u_{N_x-1}^{j+1} \end{bmatrix} = \begin{bmatrix} m & n & 0 & . & . & . & 0 & 0 & 0 \\ l & m & n & 0 & . & . & 0 & 0 & 0 \\ 0 & l & m & n & 0 & . & . & 0 & 0 \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ 0 & 0 & 0 & . & . & 0 & l & m & n \\ 0 & 0 & 0 & . & . & 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} u_1^{j} \\ u_2^{j} \\ \cdot \\ . \\ . \\ . \\ . \\ u_{N_x-2}^{j} \\ u_{N_x-1}^{j} \end{bmatrix}$$

In short form

$$M_L X^{j+1} = M_R X^j \tag{4.11}$$

Hence, the required solution is given by

$$X^{j+1} = (M_L)^{-1}\left[M_R X^j\right] \tag{4.12}$$

By the inverse operation, solution is obtained and presented in fig. 4.2. The NCS solution is compared with analytical solution and presented in fig.4.2. It shows that NCS method results are correlated with analytical solution. To check the accuracy of the NCS method, absolute error at t = 0.5 is calculated and presented in table 4.1 at different step sizes along space coordinates. It is noticed that at step size 10^{-3}, accuracy of 10^{-8} is obtained for NCS method. It is also observed from table 4.1 that if $r < 1 \text{or} r > 1$, NCS is giving stable solutions. Further NCS method is also compared with finite difference method (FDM) and results are presented graphically and absolute error tabulated. It is clear that difference between NCS method results and FDM is negligible if $r < 1$. For $r > 1$ NCS method produces better results as compared with FDM. Therefore, we conclude that NCS method is efficient and better numerical method to solve PDEs. With thisknowledge, we demonstrated the NCS method for different PDEs with different examples and presented their results graphically.

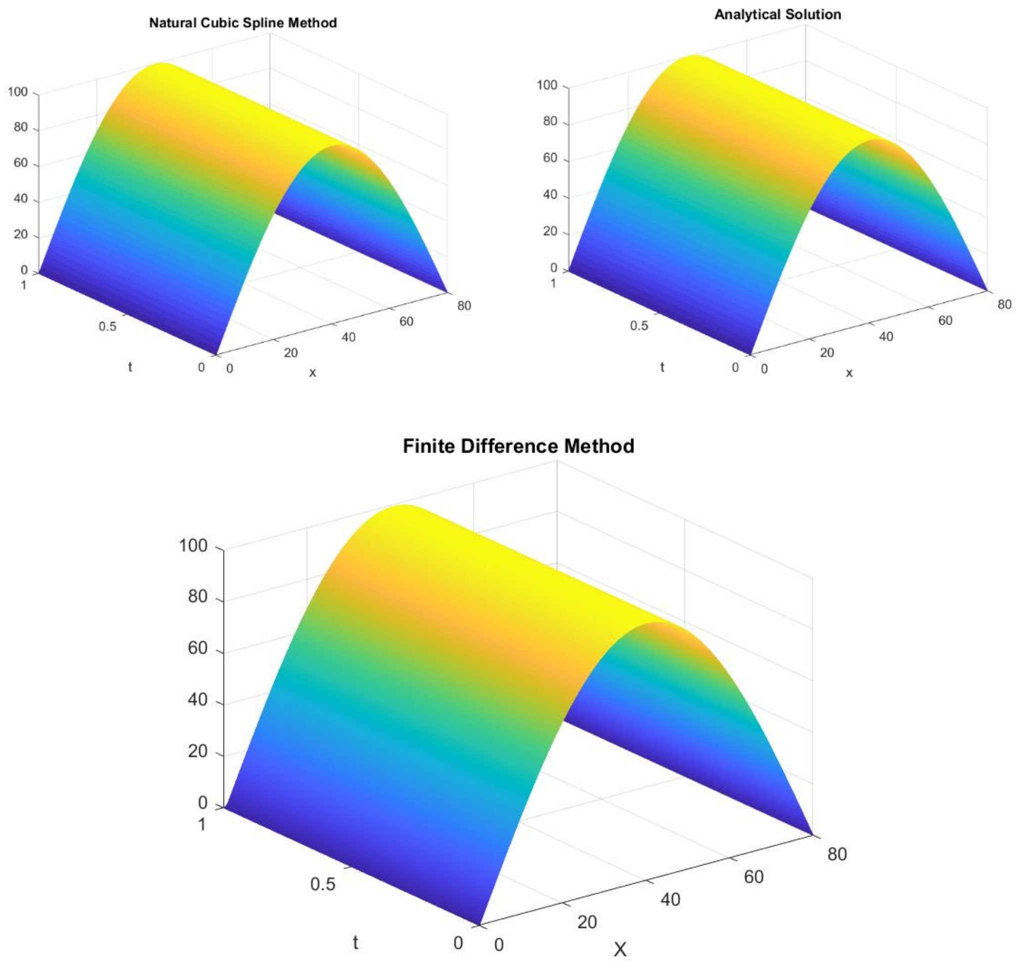

Figure 4.2: Solution of example 4.1 using NCS implicit method and analytical solution

Table 4.2: Absolute Error at t = 0.5 with 0.1 step size along t

Step size along x	r	Error	
		NCS method	Finite Difference Method
1/10	0.000156	6.3575e-04	6.3219e-04
1/50	0.003906	2.5351e-05	2.5938e-05
1/100	0.015625	6.3371e-06	6.9306e-06
1/1000	1.562500	6.3370e-08	6.5744e-07
1/2000	6.25000	1.5845e-08	6.0992e-07

EXAMPLE 4.2

Consider the PDE of the form

$$u_t = u_{xx}, \quad 0 < x < 1, t > 0 \tag{4.13}$$

subject to: $u(x,0) = 1 + x^2 + \cos \pi x,\ 0 \le x \le 1$ and $u_x(0,t) = 0,\ u_x(1,t) = 0$

The exact solution is given by $u(x,t) = 2t + x^2 + 1 + e^{-\pi^2 t}\cos(\pi x)$

EXPLICIT METHOD

By NCS explicit method (4.13) becomes

$$u_{i-1}^{j+1} + 4u_i^{j+1} + u_{i+1}^{j+1} = (1+6r)u_{i-1}^{j} + 4(1-3r)u_i^{j} + (1+6r)u_{i+1}^{j}, \quad i = 1,2,\ldots,n-1. \tag{4.14}$$

where $r = \dfrac{k}{h^2}$.

Matrix form of (4.14)

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & . & & & & & 0 & 0 \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & . & . & . & . & . & . & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & & 0 & 0 \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & l & m \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j} \\ u_{N_x}^{j} \end{bmatrix}$$

for $j = 0,1,2,\ldots,N_t$. where $l = 1+6r,\ m = 4(1-3r)$ and $n = 1+6r$.

Given condition $u_x(0,t)=0 \Rightarrow \dfrac{u_{i+1}^{j}-u_{i-1}^{j}}{2h}=0 \Rightarrow u_1^j = u_{-1}^j,\ j\geq 0.$

First equation of (4.14) becomes

$$u_{-1}^{j+1}+u_0^{j+1}+u_1^{j+1}=(1+6r)u_{-1}^j+4(1-3r)u_0^j+(1+6r)u_1^j,\ \ i=0$$
$$\Rightarrow u_0^{j+1}+2u_1^{j+1}=4(1-3r)u_0^j+(1+6r)u_1^j,$$

$$u_x(1,t)=0 \Rightarrow \frac{u_{N_x+1}^{j}-u_{N_x-1}^{j}}{2h}=0 \Rightarrow u_{N_x+1}^j=u_{N_x-1}^j,\ j\geq 0.$$

Last equation of (4.14) becomes

$$u_{n-1}^{j+1}+4u_n^{j+1}+u_{n+1}^{j+1}=(1+6r)u_{n-1}^j+4(1-3r)u_n^j+(1+6r)u_{n+1}^j,\ \ i=n$$
$$\Rightarrow 2u_{n-1}^{j+1}+4u_n^{j+1}=2(1+6r)u_{n-1}^j+4(1-3r)u_n^j,$$

Hence the above matrix reduces to

$$\begin{bmatrix} 4 & 2(1) & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & . & & & & & 0 & 0 \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & . & . & . & . & . & . & 0 & 2(1) & 4 \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} = \begin{bmatrix} m & 2n & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & & 0 & 0 \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & 2l & m \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j} \\ u_{N_x}^{j} \end{bmatrix},$$ In

short form

$$M_L X^{j+1}=M_R X^j$$

Hence the required solution

$$X^{j+1}=\left(M_L\right)^{-1}\left[M_R X^j\right]$$

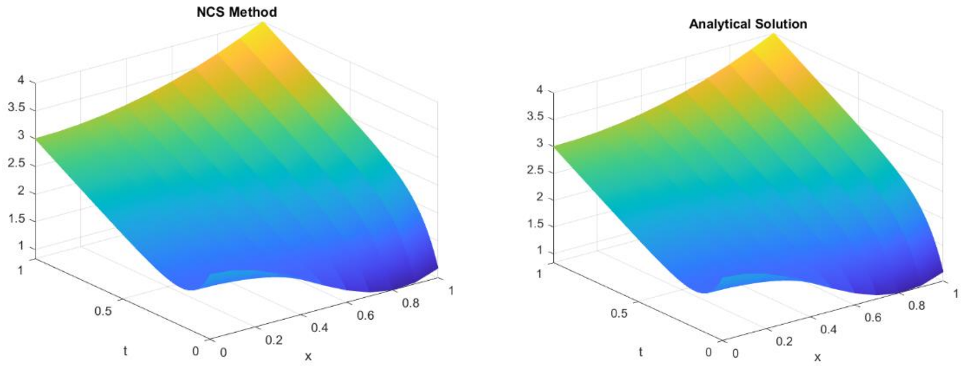

Figure 4.3: Solution of example 4.2 using NCS explicit method and analytical solution

NCS IMPLICIT METHOD

By NCS implicit method equation (4.13) becomes

$$(1-3r)u_{i-1}^{j+1}+(4+6r)u_i^{j+1}+(1-3r)u_{i+1}^{j+1}$$
$$=(1+3r)u_{i-1}^{j}+(4-6r)u_i^{j}+(1+3r)u_{i+1}^{j},\ r=\frac{k}{h^2}. \tag{4.15}$$

The above equation (4.15) is known as natural cubic spline implicit formula to solve equation (4.13)

Representing equation (4.14) into the matrix form:

$$\begin{bmatrix} a & b & c & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & a & b & c & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & a & b & c & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & a & b & c \\ 0 & 0 & . & . & . & . & . & . & 0 & a & b \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & l & m \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j} \\ u_{N_x}^{j} \end{bmatrix},$$

Where $a=1-3r, b=4+6r$ and $c=1-3r$; $l=1+3r$, $m=4-6r$ and $n=1+3r$.

for $j=0,1,2,\dots,N_t$.

Given condition $u_x(0,t)=0 \Rightarrow \dfrac{u_{i+1}^j-u_{i-1}^j}{2h}=0 \Rightarrow u_1^j=u_{-1}^j,\ j\ge 0.$

First equation of (4.15) becomes

$$(1-3r)u_{-1}^{j+1}+(4+6r)u_0^{j+1}+(1-3r)u_1^{j+1}=(1+3r)u_{-1}^{j}+(4-6r)u_0^{j}+(1+3r)u_1^{j},\ \ i=0$$
$$\Rightarrow (4+6r)u_0^{j+1}+2(1-3r)u_1^{j+1}=(4-6r)u_0^{j}+2(1+3r)u_1^{j},$$

$$u_x(1,t)=0 \Rightarrow \frac{u^j_{N_x+1}-u^j_{N_x-1}}{2h}=0 \Rightarrow u^j_{N_x+1}=u^j_{N_x-1},\ j\geq 0.$$

Last equation of (4.15) becomes

$$(1-3r)u^{j+1}_{n-1}+(4+6r)u^{j+1}_{n}+(1-3r)u^{j+1}_{n+1}=(1+3r)u^{j}_{n-1}+(4-6r)u^{j}_{n}+(1+3r)u^{j}_{n+1},\quad i=n$$

$$\Rightarrow 2(1-3r)u^{j+1}_{n-1}+(4+6r)u^{j+1}_{n}=2(1+3r)u^{j}_{n-1}+(4-6r)u^{j}_{n},$$

Hence the above matrix reduces to

$$\begin{bmatrix} b & 2c & 0 & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & a & b & c & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & a & b & c & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & a & b & c \\ 0 & 0 & . & . & . & . & . & . & 0 & 2a & b \end{bmatrix}\begin{bmatrix} u^{j+1}_0 \\ u^{j+1}_1 \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u^{j+1}_{N_x-1} \\ u^{j+1}_{N_x} \end{bmatrix}=\begin{bmatrix} m & 2n & 0 & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & 2l & m \end{bmatrix}\begin{bmatrix} u^{j}_0 \\ u^{j}_1 \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u^{j}_{N_x-1} \\ u^{j}_{N_x} \end{bmatrix}$$, In

short form

$$M_L X^{j+1}=M_R X^j$$

Hence the required solution is

$$X^{j+1}=(M_L)^{-1}\left[M_R X^j\right].$$

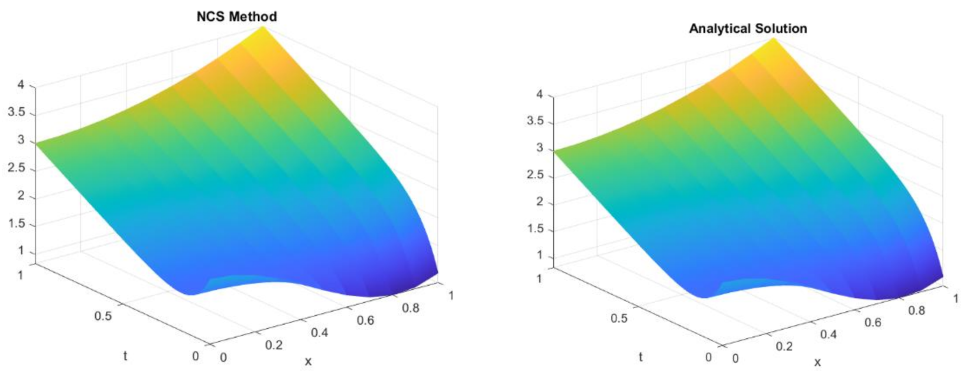

Figure 4.4: Solution of example 4.2 using NCS implicit method and analytical solution

EXAMPLE 4.3: Consider the PDE of the form

$$u_t=u_{xx}-u,\ 0<x<1,\ t>0 \tag{4.16}$$

subject to: $u(x,0)=e^{-x}+x,\ u(0,t)=1,\ u_x(1,t)=e^{-t}-e^{-1}$

The exact solution is $u(x,t) = e^{-x} + xe^{-t}$

NCS EXPLICIT METHOD

BY NCS explicit method, (4.16) becomes

$$u_{i-1}^{j+1} + 4u_i^{j+1} + u_{i+1}^{j+1} = ((1-k)+6r)u_{i-1}^j + (4(1-k)-12r)u_i^j + ((1-k)+6r)u_{i+1}^j \tag{4.17}$$

Where $r = \dfrac{k}{h^2}$. Matrix form of (4.17) reduces to

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & . & . & . & . & . & . & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & l & m \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j} \end{bmatrix}$$

where $l = (1-k)+6r,\ m = 4(1-k)-12r$ and $n = (1-k)+6r,\ j = 0,1,2,...,N_t$.

Since the conditions, $u(0,t) = 1 \Rightarrow u_0^j = 1,\ j \ge 0$, first equation of (4.17) becomes

$$u_{i-1}^{j+1} + 4u_i^{j+1} + u_{i+1}^{j+1} = ((1-k)+6r)u_{i-1}^j + (4(1-k)-12r)u_i^j + ((1-k)+6r)u_{i+1}^j,\ \ i = 1$$
$$\Rightarrow 4u_1^{j+1} + u_2^{j+1} + 1 = (4(1-k)-12r)u_1^j + ((1-k)+6r)u_2^j + 1,$$

and $u_x(1,t) = e^{-t} - e^{-1} \Rightarrow, u_{N_x+1}^j = u_{N_x-1}^j + 2h(e^{-t} - e^{-1}),\ j \ge 0.$

last equation in (4.17) becomes

$$u_{n-1}^{j+1} + 4u_n^{j+1} + u_{n+1}^{j+1} = ((1-k)+6r)u_{n-1}^j + (4(1-k)-12r)u_n^j + ((1-k)+6r)u_{n+1}^j,\ \ i = n$$
$$\Rightarrow 2u_{n-1}^{j+1} + 4u_n^{j+1} + 2h(e^{-t^j} - e^{-1}) = 2((1-k)+6r)u_{n-1}^j + (4(1-k)-12r)u_n^j + 2h(e^{-t^j} - e^{-1})$$

Hence the above matrix reduces to

$$\begin{bmatrix} 4 & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ 1 & 4 & 1 & 0 & . & . & . & . & 0 & 0 \\ 0 & 1 & 4 & 1 & . & . & . & . & 0 & 0 \\ & & & & . & & & & & \\ & & & & . & & & & & \\ & & & & . & & & & & \\ & & & & . & & & & & \\ & & & & . & & & & & \\ 0 & 0 & 0 & 0 & . & . & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & . & . & 0 & 0 & 2(1) & 4 \end{bmatrix} \begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 2h(e^{-t}-e^{-1}) \end{bmatrix} =$$

$$\begin{bmatrix} m & n & 0 & 0 & . & . & . & . & 0 & 0 \\ l & m & n & 0 & . & . & . & . & 0 & 0 \\ 0 & l & m & n & . & . & . & . & 0 & 0 \\ & & & & . & & & & & \\ & & & & . & & & & & \\ & & & & . & & & & & \\ & & & & . & & & & & \\ & & & & . & & & & & \\ 0 & 0 & 0 & 0 & . & . & 0 & l & m & n \\ 0 & 0 & 0 & 0 & . & . & 0 & 0 & 2(l) & m \end{bmatrix} \begin{bmatrix} u_1^{j} \\ u_2^{j} \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j} \\ u_{N_x}^{j} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 2h(e^{-t}-e^{-1}) \end{bmatrix}$$

$$\Rightarrow M_L X^{j+1} + C_L = M_R X^j + C_R$$

$$\Rightarrow X^{j+1} = \left(M_L\right)^{-1}\left[M_R X^j + C_R - C_L\right]$$

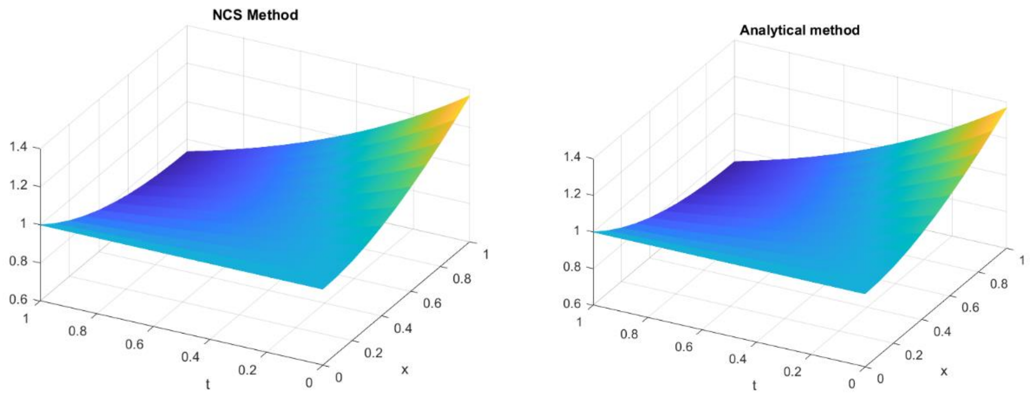

Figure 4.5: Solution of example 4.3 using NCS explicit method and analytical solution

NCS IMPLICIT METHOD

BY NCS explicit method (4.16) becomes

$$(1-3r)u_{i-1}^{j+1}+(4+6r)u_i^{j+1}+(1-3r)u_{i+1}^{j+1}$$
$$=((1-k)+3r)u_{i-1}^{j}+(4(1-k)-6r)u_i^{j}+((1-k)+3r)u_{i+1}^{j},\ r=\frac{k}{h^2}. \tag{4.18}$$

The above equation (4.18) is known as natural cubic spline implicit formula to solve equation (4.18)

Representing (4.18) in matrix form we have

$$\begin{bmatrix} a & b & c & 0 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & a & b & c & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 0 & a & b & c & . & & & & & 0 & 0 \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & a & b & c \\ 0 & 0 & . & . & . & . & . & . & 0 & a & b \end{bmatrix}\begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \cdot \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix}=\begin{bmatrix} l & m & n & 0 & 0 & . & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & & 0 & 0 \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ & & & & & . & & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & l & m \end{bmatrix}\begin{bmatrix} u_0^{j} \\ u_1^{j} \\ \cdot \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j} \end{bmatrix}$$

Where

$a=1-3r,\ b=4+6r$ and $c=1-3r$

And $l=(1-k)+3r,\ m=4(1-k)-6r,\ n=(1-k)+3r,\ j=0,1,2,...,N_t.$

Since the conditions, $u(0,t)=1\Rightarrow u_0^j=1,\ j\geq 0$, first equation of (4.18) becomes

$$(1-3r)u_{i-1}^{j+1}+(4+6r)u_i^{j+1}+(1-3r)u_{i+1}^{j+1}$$
$$=((1-k)+3r)u_{i-1}^{j}+(4(1-k)-6r)u_i^{j}+((1-k)+3r)u_{i+1}^{j},\ \ i=1$$
$$\Rightarrow (4+6r)u_1^{j+1}+(1-3r)u_2^{j+1}+1=(4(1-k)-6r)u_1^{j}+((1-k)+3r)u_2^{j}+1,$$

and $u_x(1,t)=e^{-t}-e^{-1}\Rightarrow, u_{N_x+1}^{j}=u_{N_x-1}^{j}+2h(e^{-t^j}-e^{-1}),\ j\geq 0.$

last equation in (4.18) becomes

$$(1-3r)u_{i-1}^{j+1}+(4+6r)u_i^{j+1}+(1-3r)u_{i+1}^{j+1}$$
$$=((1-k)+3r)u_{i-1}^{j}+(4(1-k)-6r)u_i^{j}+((1-k)+3r),\ \ i=n$$
$$\Rightarrow 2(1-3r)u_{n-1}^{j+1}+(4+6r)u_n^{j+1}+2h(e^{-t^j}-e^{-1})$$
$$=2((1-k)+3r)u_{n-1}^{j}+(4(1-k)-6r)u_n^{j}+2h(e^{-t^j}-e^{-1})$$

Hence the above matrix reduces to

$$
\begin{bmatrix}
b & c & 0 & 0 & . & . & . & . & 0 & 0\\
a & b & c & 0 & . & . & . & . & 0 & 0\\
0 & a & b & c & . & . & . & . & 0 & 0\\
 & & & & & . & & & & \\
 & & & & & . & & & & \\
 & & & & & . & & & & \\
 & & & & & . & & & & \\
 & & & & & . & & & & \\
0 & 0 & 0 & 0 & . & . & 0 & a & b & c\\
0 & 0 & 0 & 0 & . & . & 0 & 0 & 2(a) & b
\end{bmatrix}
\begin{bmatrix}
u_1^{j+1}\\ u_2^{j+1}\\ .\\ .\\ .\\ .\\ .\\ .\\ u_{N_x-1}^{j+1}\\ u_{N_x}^{j+1}
\end{bmatrix}
+
\begin{bmatrix}
1\\ 0\\ .\\ .\\ .\\ .\\ .\\ .\\ 0\\ 2h(e^{-t}-e^{-1})
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
m & n & 0 & 0 & . & . & . & . & 0 & 0\\
l & m & n & 0 & . & . & . & . & 0 & 0\\
0 & l & m & n & . & . & . & . & 0 & 0\\
 & & & & & . & & & & \\
 & & & & & . & & & & \\
 & & & & & . & & & & \\
 & & & & & . & & & & \\
 & & & & & . & & & & \\
0 & 0 & 0 & 0 & . & . & 0 & l & m & n\\
0 & 0 & 0 & 0 & . & . & 0 & 0 & 2(l) & m
\end{bmatrix}
\begin{bmatrix}
u_1^{j}\\ u_2^{j}\\ .\\ .\\ .\\ .\\ .\\ .\\ u_{N_x-1}^{j}\\ u_{N_x}^{j}
\end{bmatrix}
+
\begin{bmatrix}
1\\ 0\\ .\\ .\\ .\\ .\\ .\\ .\\ 0\\ 2h(e^{-t}-e^{-1})
\end{bmatrix}
$$

$$\Rightarrow M_L X^{j+1} + C_L = M_R X^j + C_R$$

$$\Rightarrow X^{j+1} = \left(M_L\right)^{-1}\left[M_R X^j + C_R - C_L\right]$$

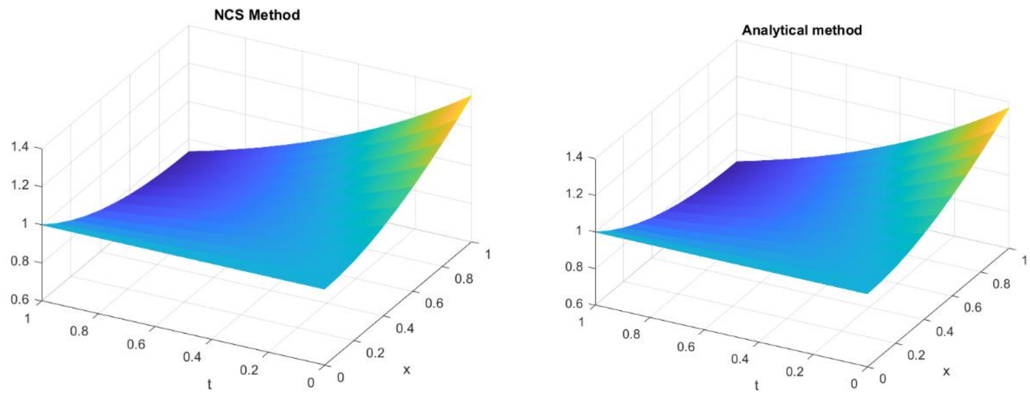

Figure 4.6: Solution of example 4.3 using NCS implicit method and analytical solution

EXAMPLE 4.4: Consider PDE of the form:

$$u_t = u_{xx} + 2tu,\ 0 < x < 1,\ t > 0 \tag{4.19}$$

Subject to: $u(x,0)=e^{x}$, $u(0,t)=e^{t+t^2}$, $u(1,t)=e^{1+t+t^2}$

The exact solution is $u(x,t)=e^{x+t+t^2}$

NCS EXPLICIT METHOD

By NCS explicit formula (4.19) becomes

$$u_{i-1}^{j+1}+4u_{i}^{j+1}+u_{i+1}^{j+1}=((1+2kt^{j})+6r)u_{i-1}^{j}+(4(1+2kt^{j})-12r)u_{i}^{j} \\ +((1+2kt^{j})+6r)u_{i+1}^{j} \tag{4.20}$$

The matrix form of (4.20) reduces to

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & . & . & . & . & . & . & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix}=\begin{bmatrix} l & m & n & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & l & m \end{bmatrix}\begin{bmatrix} u_0^{j} \\ u_1^{j} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j} \end{bmatrix} \text{ where}$$

$l=1+2kt^{j}+6r,\ m=4(1+2kt^{j})-12r$ and $n=1+2kt^{j}+6r,\ j=0,1,2,\ldots,N_t.$

Since the condition, $u(0,t)=e^{t+t^2}\Rightarrow u_0^{j}=e^{t+t^2},\ j\geq 0,$ first equation of (4.20) becomes

$$u_{i-1}^{j+1}+4u_{i}^{j+1}+u_{i+1}^{j+1} \\ =((1+2kt^{j})+6r)u_{i-1}^{j}+(4(1+2kt^{j})-12r)u_{i}^{j}+((1+2kt^{j})+6r)u_{i+1}^{j},\ i=1$$

$$\Rightarrow 4u_1^{j+1}+u_2^{j+1}+e^{t+t^2}=(4(1+2kt^{j})-12r)u_1^{j}+((1+2kt^{j})+6r)u_2^{j}+e^{t+t^2}$$

another condition, $u(1,t)=0\Rightarrow u_{N_x}^{j}=e^{1+t+t^2},\ j\geq 0,$ last equation of (4.19) becomes

$$u_{i-1}^{j+1}+4u_{i}^{j+1}+u_{i+1}^{j+1} \\ =((1+2kt^{j})+6r)u_{i-1}^{j}+(4(1+2kt^{j})-12r)u_{i}^{j}+((1+2kt^{j})+6r)u_{i+1}^{j},\ i=n-1$$

$$\Rightarrow u_{n-2}^{j+1}+4u_{n-1}^{j+1}+e^{1+t^{j}+t^{j2}}=(4(1+2kt^{j})-12r)u_{n-1}^{j}+((1+2kt^{j})+6r)u_{n-2}^{j}+e^{1+t^{j}+t^{j2}}$$

The above matrix reduces to

$$
\begin{bmatrix}
4 & 1 & 0 & . & . & . & 0 & 0 & 0 \\
1 & 4 & 1 & 0 & . & . & 0 & 0 & 0 \\
0 & 1 & 4 & 1 & 0 & . & . & 0 & 0 \\
 & & & & . & & & & \\
 & & & & . & & & & \\
 & & & & . & & & & \\
 & & & & . & & & & \\
0 & 0 & 0 & . & . & 0 & 1 & 4 & 1 \\
0 & 0 & 0 & . & . & 0 & 0 & 1 & 4
\end{bmatrix}
\begin{bmatrix}
u_1^{j+1} \\ u_2^{j+1} \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-2}^{j+1} \\ u_{N_x-1}^{j+1}
\end{bmatrix}
+
\begin{bmatrix}
e^{t+t^2} \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ e^{1+t+t^2}
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
m & n & 0 & . & . & . & 0 & 0 & 0 \\
l & m & n & 0 & . & . & 0 & 0 & 0 \\
0 & l & m & n & 0 & . & . & 0 & 0 \\
 & & & & . & & & & \\
 & & & & . & & & & \\
 & & & & . & & & & \\
 & & & & . & & & & \\
0 & 0 & 0 & . & . & 0 & l & m & n \\
0 & 0 & 0 & . & . & 0 & 0 & l & m
\end{bmatrix}
\begin{bmatrix}
u_1^{j} \\ u_2^{j} \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-2}^{j} \\ u_{N_x-1}^{j}
\end{bmatrix}
+
\begin{bmatrix}
e^{t+t^2} \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ e^{1+t+t^2}
\end{bmatrix}
$$

$$\Rightarrow M_L X^{j+1} + C_L = M_R X^j + C_R$$

$$\Rightarrow X^{j+1} = \left(M_L\right)^{-1}\left[M_R X^j + C_R - C_L\right].$$

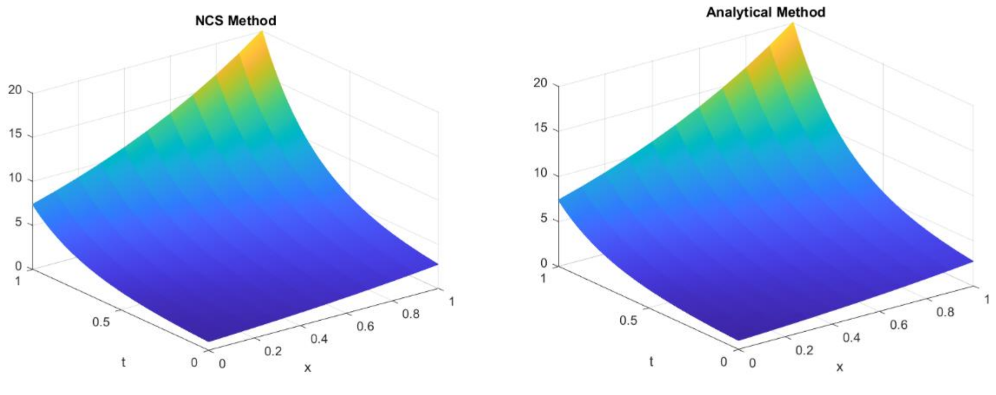

Figure 4.7: Solution of example 4.4 using NCS explicit method and analytical solution

NCS IMPLICIT METHOD

By using NCS implicit formula (4.19) becomes

$$
\begin{aligned}
&(1-3r)u_{i-1}^{j+1} + (4+6r)u_i^j + (1-3r)u_{i+1}^{j+1} \\
&\quad = (1+2kt^j+3r)u_{i-1}^j + (4(1+2kt^j)-6r)u_i^j + ((1+2kt^j)+3r)u_{i+1}^j, \ r = \frac{k}{h^2}.
\end{aligned}
\tag{4.21}
$$

The above equation (4.21) is known as natural cubic spline implicit formula to solve equation (4.19).

Representing (4.21) in matrix form we have

$$\begin{bmatrix} a & b & c & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & a & b & c & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & a & b & c & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & a & b & c \\ 0 & 0 & . & . & . & . & . & . & 0 & a & b \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \cdot \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & l & m & n & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & l & m & n & . & & & & 0 & 0 \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ & & & & & . & & & & & \\ 0 & 0 & . & . & . & . & 0 & 0 & l & m & n \\ 0 & 0 & . & . & . & . & . & . & 0 & l & m \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ \cdot \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j} \end{bmatrix}$$

Where $a = 1-3r, b = 4+6r$ and $c = 1-3r$;

$l = 1+2kt^j + 6r,\ m = 4(1+2kt^j) - 12r$ and $n = 1+2kt^j + 6r,\ j = 0,1,2,...,N_t.$

Since $u(0,t) = e^{t+t^2} \Rightarrow u_0^j = e^{t+t^2},\ j \geq 0.$ and $u(1,t) = 0 \Rightarrow u_{N_x}^j = e^{1+t+t^2},\ j \geq 0.$

First equation of (4.21) becomes

$$\begin{aligned}(1-3r)u_{i-1}^{j+1} &+ (4+6r)u_i^j + (1-3r)u_{i+1}^{j+1} \\ &= (1+2kt^j + 3r)u_{i-1}^j + (4(1+2kt^j) - 6r)u_i^j + ((1+2kt^j)+3r)u_{i+1}^j,\ i = 1\end{aligned}$$

$$\Rightarrow (4+6r)u_1^j + (1-3r)u_2^{j+1} + e^{t+t^2} = (4(1+2kt^j) - 6r)u_1^j + ((1+2kt^j)+3r)u_2^j + e^{t+t^2}$$

$$u(1,t) = 0 \Rightarrow u_{N_x}^j = e^{1+t+t^2},\ j \geq 0.$$

Last equation of (4.21) becomes

$$\begin{aligned}(1-3r)u_{i-1}^{j+1} &+ (4+6r)u_i^j + (1-3r)u_{i+1}^{j+1} \\ &= (1+2kt^j + 3r)u_{i-1}^j + (4(1+2kt^j) - 6r)u_i^j + ((1+2kt^j)+3r)u_{i+1}^j,\ i = n-1\end{aligned}$$

$$\Rightarrow (4+6r)u_{n-1}^j + (1-3r)u_{n-2}^{j+1} + e^{1+t^j+t^{j2}} = (4(1+2kt^j) - 6r)u_{n-2}^j + ((1+2kt^j)+3r)u_{n-1}^j + e^{1+t^j+t^{j^2}}$$

Hence the above matrix reduces to

$$\begin{bmatrix} b & c & 0 & . & . & . & 0 & 0 & 0 \\ a & b & c & 0 & . & . & 0 & 0 & 0 \\ 0 & a & b & c & 0 & . & . & 0 & 0 \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ 0 & 0 & 0 & . & . & 0 & a & b & c \\ 0 & 0 & 0 & . & . & 0 & 0 & a & b \end{bmatrix} \begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-2}^{j+1} \\ u_{N_x-1}^{j+1} \end{bmatrix} + \begin{bmatrix} e^{t+t^2} \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ e^{1+t+t^2} \end{bmatrix} =$$

$$\begin{bmatrix} m & n & 0 & . & . & . & 0 & 0 & 0 \\ l & m & n & 0 & . & . & 0 & 0 & 0 \\ 0 & l & m & n & 0 & . & . & 0 & 0 \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ & & & & . & & & & \\ 0 & 0 & 0 & . & . & 0 & l & m & n \\ 0 & 0 & 0 & . & . & 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} u_1^{j} \\ u_2^{j} \\ . \\ . \\ . \\ . \\ . \\ u_{N_x-2}^{j} \\ u_{N_x-1}^{j} \end{bmatrix} + \begin{bmatrix} e^{t+t^2} \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ e^{1+t+t^2} \end{bmatrix}$$

In short

$$M_L X^{j+1} + C_L = M_R X^j + C_R$$

Hence the required solution is

$$X^{j+1} = \left(M_L\right)^{-1}\left[M_R X^j + C_R - C_L\right].$$

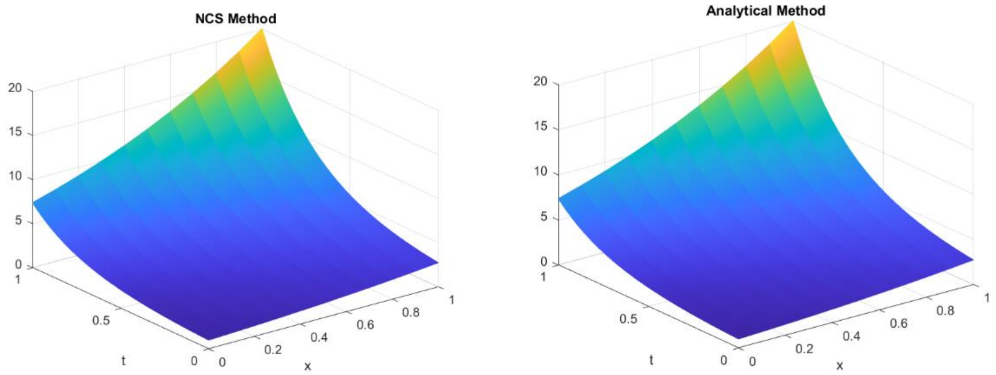

Figure 4.8: Solution of example 4.4 using NCS implicit method and analytical solution

6. CONCLUSION

In this Paper, NCS method is employed to solve parabolic PDE. In detailed procedure is explained for NCS method for two different types such as explicit and implicit. Considered different examples to demonstrate the developed NCS method. The results for all examples are presented graphically. Accuracy of NCS method is calculated through absolute error considering analytical solutions. It is worth to mention that for NCS implicit method produced better and more accurate solutions compare to explicit. Further NCS method is also compared with finite difference method and results are presented graphically and absolute error tabulated. It is clear that difference between NCS method results and FDM is negligible if $r<1$. For $r>1$ NCS method produces better results as compared with FDM. Therefore, we conclude that NCS method is efficient and better numerical method to solve PDEs. With this knowledge, we demonstrated the NCS method for different PDEs with different examples and presented their results graphically.

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