On The Existence and the Uniqueness of Solutions of the Fredholm Integral Equations of the Second Kind on the Contour

Mostefa Nadir

Laboratory of Pure and Applied Mathematics, Department of Mathematics, University of Msila,
28000 ALGERIA
mostefanadir@yahoo.fr

ABSTRACT:

The basic goal of this work is to obtain the conditions of existence and uniqueness of the solution of the integral equations of the second kind. Noting that, except the Banach's theorem where the norm of the integral operator must be less than unity, the existence and uniqueness of the solution of the integral equations of the second kind remain an open question

KEY WORDS AND PHRASES: Singular integral operator, Fredholm equation, Algebras theory.

INTRODUCTION

We try to look for the existence and the uniqueness of the solution of the Fredholm integral equation of the second kind on the contour L

$$\varphi(t_0) - \int_L k(t, t_0)\varphi(t)\,dt = f(t_0), \qquad (1)$$

where L is a closed curve in the complex plane, does not intersect itself and is given by an equation

$$t = t(s),\ a \le s \le b,$$

with the function t(s) is continuously differentiable and its derivative is everywhere different from zero.
Noting that, in the applications of these equations to different problems,
the following relations are known to play a basic part

- The homogeneous equation

$$\varphi(t_0) - \int_L k(t, t_0)\varphi(t)\,dt = 0, \qquad (2)$$

has only a finite number of linearly independent solutions.

- The adjoint homogeneous equation

$$\psi(t_0) - \int_L k^*(t_0, t)\psi(t)\,dt = 0, \qquad (3)$$

have the same number of linearly independent solutions [Nadir, Lakehali 2006; Nadir 2004; Okecha 1986].

- The non-homogeneous equations (1) is solvable for any second member f if and only if the adjoint homogeneous equation (3) or the equation homogeneous (2) has non solution different from zero [Atkinson, (1967); Yucheng Liu, 2009; Akyüz-Daşcıoğlu A. and Çerdik Y. H. (2006); Baboliana et al (2008); Hadizadeh et al (2005)].

- The solvability of the non-homogeneous equation (1) is given by the necessary and sufficient conditions

$$\int_L f(t)\psi_k(t)\,dt = 0, \qquad (4)$$

where the functions k(t) form a complete system of linearly independent solutions of the adjoint homogeneous solutions (3) (Muskhelishvili,1968).

Lemme1 (Hadizadeh et al 2005).
The integral operator S defined by

$$S\varphi(t_0) = \frac{1}{\pi i}\int_L \frac{\varphi(t)}{t - t_0}\,dt, \qquad (5)$$

is bounded in all Hölder spaces $C^{\alpha}(L)$, $0 < \alpha < 1$; whenever the function φ(t) satisfies the Hölder condition on the curve L:

$$|\varphi(t) - \varphi(t_0)| \le M\,|t - t_0|^{\alpha}, \quad 0 < \alpha \le 1. \qquad (6)$$

Let a(t) be a function in $C^{\alpha}(L)$: Then the commentator

$$(aS - Sa)\varphi = \frac{1}{\pi i}\int_L \frac{a(t_0) - a(t)}{t - t_0}\varphi(t)\,dt, \qquad (7)$$

is compact from $C^{\alpha}(L)$ into $C^{\alpha}(L)$

In fact, since the function a(t) $\in C^{\alpha}(L)$, then the kernel $\frac{a(t) - a(t_0)}{t - t_0}$ has a weak singularity and defines a compact integral operator in all spaces $C^{\alpha}(L)$

COROLLARY

The property of the compactness is enjoyed by the more general operator

$$\int_L \frac{p(t, t_0) - q(t, t_0)}{t - t_0}\varphi(t)\,dt, \qquad (8)$$

if the functions $p(t; t0)$ and $q(t; t0)$ are of Hölder class in both variables.

In particular the commutator operator

$$(aT - Ta)\varphi,$$

where the operator T is given by

$$T\varphi(t_0) = \frac{1}{\pi i}\int_L \frac{k(t,t_0)}{t-t_0}\varphi(t)\,dt,$$

is also compact from $C^{\alpha}(L)$ into $C^{\alpha}(L)$

MAIN RESULTS:

THEOREM

let $k(t; t0)$ *be a function in* $C^{\alpha}(L)$ satisfi*es the Hölder condition for both variables, then the equation* (1)

$$\varphi(t_0) - \int_L k(t,t_0)\varphi(t)\,dt = f(t_0),$$

admits a unique solution in the space $C^{\alpha}(L)$ *for all second member* $f(t)$ *in* $C^{\alpha}(L)$

PROOF

It is clear to see that, the compact operator integral

$$A\varphi(t_0) = \int_L k(t,t_0)\varphi(t)\,dt,$$

has the commutator operator representation

$$[a,T]\varphi = (aT - Ta)\varphi,$$

with the function $a(s) = s$ for all s □ L:

Also, it is known that, the unit element $I\varphi(t) = \varphi(t)$ of a Banach algebra $C^{\alpha}(L)$ is not a commutator

$$AB - BA;$$

of two elements A and B in $C^{\alpha}(L)$

Indeed, if

$$I = AB - BA;$$

then the spectrum relation gives

$$sp(AB) = 1 + sp(BA);$$

which is not consistent with the following one

$$sp(AB) \cup \{0\} = sp(BA) \cup \{0\}$$

Therefore,

$$\int_L k(t,t_0)\varphi(t)dt \neq \varphi(t_0), \quad \text{for all } \varphi \in C^\alpha(L).$$

So, the homogeneous equation has only the trivial solution in C $^\alpha$ (L) and the result can be obtained.

REFERENCES

Nadir M., Lakehali B., (2007) "*An approximation of singular integrals FEN DERGISI (E-DERGI) 2(2), 236-240.(2007).*

Nadir M., (2004) "*On the approximation of singular integrals of Cauchy types*", in Proceeding Dynamic System and Applications (2004)

Okecha G. E. (1986) "*Evaluating Singular Integrals Of Cauchy Type Using A Modified Gaussian Rule*", Int. J. of Computer Mathematics, 19, 85 - 92

Atkinson, K. E., (1967) "*The numerical solution of Fredholm integral equations of the second kind*" SIAM J. 4, 337—348.

Yucheng Liu, 2009 " *Application of the Chebyshev polynomial in solving Fredholm integral equations*" Mathematical and Computer Modelling, 50, 465--469.

Akyüz-Daşcıoğlu A. and Çerdik Y. H. (2006) "*An approximation method for the solution of nonlinear integral equations*" Applied Mathematics and Computation, 174, 619--629.

Baboliana E., Abbasbandya S. and Fattahzadeha F. (2008) "*A numerical method for solving a class of functional and two dimensionalintegral equations*" Applied Mathematics and Computation, 198, 35-43.

W.Pogorzelski, (1966) Integral equations and their applictions volume 1, Polish scientific Publishers Warszawa

Nadir M., J. of the Association of Arab Univ. for Basic and Applied Science, Vol. 7, 2009, 110-114

حول حلول معادلات فريد هو لم التكاملية من النوع الثاني

مصطفى نادر

مختبر الرياضيات البحتة والتطبيقية ، قسم الرياضيات ، جامعة المسيلة الجزائر
mostefanadir@yahoo.fr

الملخص

الهدف الأساسي من هذا البحث هو الحصول على شروط الوجود و الوحدانية لحلول المعادلات التكاملية لفريدولم من الدرجة الثانية. بأعتباراً أنه لا يوجد سوى مبرهنة بناخ أين يشترط نظيم التكامل يكون أقل تماما من الوحدة عدا ذلك تكون يبقى حل الوجود و الوحدانية لهذا النوع من المعادلات مطروحا حتى يومنا هذا.