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الحلول التقريبية للمعادلات من نوع فولترا وفريدهولم باستخدام حزمه محسنه من الدوال

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الملخص:

فى هذا البحث لقد تم دراسة بعض حلول معادلات من نوع فولترا وفريدهولوم غير الخطية من النوع الأول. بإستخدام مجموعة من المعادلات ثنائية البعد وتأثيراتها التكاملية، فإن معادلات فولترا وفريدهولوم غير الخطية من النوع الأول يمكن أن تتحول إلى معادلات خطية. كما تمت دراسة بعض الأمثلة التى تبين التقارب والتباعد لهذه الطريقة، ومن خلالها تبين أن الطريقة العددية لها بعض المميزات بدرجة عالية من الدقه.

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ORIGINAL ARTICLE

Approximate solutions for mixed nonlinear Volterra–Fredholm type integral equations via modified block-pulse functions

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Mixed nonlinear Volterra–Fredholm type integral equations;
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Operational matrix

Abstract In this article a robust approach for solving mixed nonlinear Volterra–Fredholm type integral equations of the first kind is investigated. By using the modified two-dimensional block-pulse functions (M2D-BFs) and their operational matrix of integration, first kind mixed nonlinear Volterra–Fredholm type integral equations can by reduced to a nonlinear system of equations. The coefficients matrix of this system is a block matrix with lower triangular blocks. Some theorems are included to show the convergence and advantage of this method. Numerical results show that the approximate solutions have a good degree of accuracy.

1. Introduction

In this paper we applied the direct method for solving mixed nonlinear Volterra–Fredholm type integral equations of the first kind of the form:

$$\int_0^x \int_\Omega G(x,y,s,t,u(s,t))dtds = f(x,y); \quad (x,y) \in [0,1) \times \Omega, \tag{1}$$

where $u(s,t)$ is an unknown function, $f(x,y)$ and $G(x,y,s,t,u(s,t))$ are analytical function on $[0,1) \times \Omega$ and $[0,1) \times \Omega^4$, respectively, where Ω is a close subset on $\mathbb{R}^d (d = 1,2,3)$. Existence and uniqueness results for Eq. (1) may be found in (Diekmann, 1978; Pachpatte, 1978; Thieme, 1977).

Equation of type (1) often arise from the mathematical modeling of the spreading, in space and time, of some contagious disease in a population living in a habitat Ω (Diekmann, 1978; Thieme, 1977), in the theory of nonlinear parabolic boundary value problems (Pachpatte, 1978), and in many physical and biological models.

The literature on numerical methods for solving Eq. (1) mainly consists of projection methods, collocation methods, the trapezoidal Nyström method, Adomain decomposition method, He's homotopy perturbation method and the two-dimensional block-pulse functions (Adomian, 1990, 1994; Adomian and Rach, 1992; Biazar et al., 2011; Brunner, 1990; Cardone et al., 2006; Cherruault et al., 1992; Guoqiang, 1995; Hacia, 1996; Kauthen, 1989; Maleknejad and Fadaei Yami, 2006; Maleknejad and Hadizadeh, 1999; Maleknejad and Mahdiani, 2011; Wazwaz, 2006; Yee, 1993).

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Assume now that:

$$G(x,y,s,t,u(s,t)) = k(x,y,s,t)[u(s,t)]^p, \tag{2}$$

where p is a positive integer. In the present paper, we apply a modification of block-pulse functions (Maleknejad and Rahimi, 2011), to solve the mixed nonlinear Volterra–Fredholm type integral Eq. (1) with Eq. (2).

2. M2D-BFs and their properties

Definition 1. An $(m+1)^2$-set of M2D-BFs consists of $(m+1)^2$ functions which are defined over district $D=[0,1)\times[0,1)$ as follows:

$$\phi_{i_1,i_2}(x,y) = \begin{cases} 1 & (x,y)\in D_{i_1,i_2}, \quad i_1,i_2 = 0(1)m, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

where $D_{i_1,i_2} = \{(x,y)|x\in I_{i_1,\varepsilon}, y\in I_{i_2,\varepsilon}\}$, and

$$I_{\alpha,\varepsilon} = \begin{cases} [0,h-\varepsilon) & \alpha=0, \\ [\alpha h-\varepsilon,(\alpha+1)h-\varepsilon) & \alpha=1(1)m, \\ [1-\varepsilon,1) & \alpha=m. \end{cases} \tag{4}$$

where m is an arbitrary positive integer, and $h=\frac{1}{m}$.

Since, each M2D-BF takes only one value in its subregion, the M2D-BFs can be expressed by the two modified one-dimensional block-pulse functions (M1D-BFs):

$$\phi_{i_1,i_2}(x,y) = \phi_{i_1}(x)\phi_{i_2}(y), \tag{5}$$

where $\phi_{i_1}(x)$ and $\phi_{i_2}(y)$ are the M1D-BFs related to variables x and y, respectively. The M2D-BFs are disjointed with each other:

$$\phi_{i_1,i_2}(x,y)\phi_{j_1,j_2}(x,y) = \begin{cases} \phi_{i_1,i_2}(x,y) & i_1=j_1, i_2=j_2, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

and are orthogonal with each other:

$$\int_0^1\int_0^1 \phi_{i_1,i_2}(x,y)\phi_{j_1,j_2}(x,y)dydx = \begin{cases} \triangle(I_{i_1,\varepsilon})\triangle(I_{i_2,\varepsilon}) & i_1=j_1, i_2=j_2, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

where $(x,y)\in D$, $i_1,i_2,j_1,j_2 = 0(1)m$ and $\triangle(I_{i_1,\varepsilon})$ and $\triangle(I_{i_2,\varepsilon})$ are length of intervals $I_{i_1,\varepsilon}$ and $I_{i_2,\varepsilon}$, respectively.

2.1. Vector forms

Consider the first $(m+1)^2$ terms of M2D-BFs and write them concisely as $(m+1)^2$-vector:

$$\Phi_{m,\varepsilon}(x,y) = [\phi_{0,0}(x,y),\ldots,\phi_{0,m}(x,y),\ldots,\phi_{m,0}(x,y),\ldots, \phi_{m,m}(x,y)]^T;\ (x,y)\in D. \tag{8}$$

Whence Eqs. (6) and (8) implies that:

$$\Phi_{m,\varepsilon}(x,y)\Phi^T_{m,\varepsilon}(x,y) = \begin{pmatrix} \phi_{0,0}(x,y) & 0 & \ldots & 0 \\ 0 & \phi_{0,1}(x,y) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \phi_{m,m}(x,y) \end{pmatrix}_{(m+1)^2\times(m+1)^2}. \tag{9}$$

Now suppose that X be a $(m+1)^2$-vector. Hence by using Eq. (9) we obtain:

$$\Phi_{m,\varepsilon}(x,y)\Phi^T_{m,\varepsilon}(x,y)X = \widetilde{X}\Phi_{m,\varepsilon}(x,y), \tag{10}$$

where $\widetilde{X} = diag(X)$ is a $(m+1)^2\times(m+1)^2$ diagonal matrix.

2.2. M2D-BFs expansions

A function $f(x,y)$ defined over district $L^2(D)$ may be expanded by the M2D-BFs as:

$$f(x,y) \simeq f_{m,\varepsilon}(x,y) = \sum_{i_1=0}^m\sum_{i_2=0}^m f_{i_1,i_2}\phi_{i_1,i_2}(x,y) = F^T_{m,\varepsilon}\Phi_{m,\varepsilon}(x,y) = \Phi^T_{m,\varepsilon}(x,y)F_{m,\varepsilon}, \tag{11}$$

where $F_{m,\varepsilon}$ is an $(m+1)^2\times 1$ vector given by

$$F_{m,\varepsilon} = [f_{0,0},\ldots,f_{0,m},\ldots,f_{m,0},\ldots,f_{m,m}]^T, \tag{12}$$

and $\Phi_{m,\varepsilon}(x,y)$ is defined in Eq. (8), and f_{i_1,i_2}, are obtained as:

$$f_{i_1,i_2} = \frac{1}{\triangle(I_{i_1,\varepsilon})\triangle(I_{i_2,\varepsilon})}\int_{I_{i_1,\varepsilon}}\int_{I_{i_2,\varepsilon}} f(x,y)dydx. \tag{13}$$

Similarly a function of four variables, $k(x,y,s,t)$, on district $L^2(D\times D)$ may be approximated with respect to M2D-BFs such as:

$$k(x,y,s,t) \simeq \Phi^T_{m,\varepsilon}(x,y)K_{m,\varepsilon}\Phi_{m,\varepsilon}(s,t), \tag{14}$$

where $\Phi_{m,\varepsilon}(x,y)$ and $\Phi_{m,\varepsilon}(s,t)$ are M2D-BFs vector of dimension $(m+1)^2$, and $K_{m,\varepsilon}$ is the $(m+1)^2\times(m+1)^2$ M2D-BFs coefficients matrix.

3. Convergence analysis

In this section, we show that the given method in the previous sections, is convergent and its order of convergence is $O(\frac{1}{km})$. For our purposes we will need the following theorems.

Theorem 1. *Let*

$$f_{m,\varepsilon}(x,y) = \sum_{i_1=0}^m\sum_{i_2=0}^m f_{i_1,i_2}\phi_{i_1,i_2}(x,y),$$

and

$$f_{i_1,i_2} = \frac{1}{\triangle(I_{i_1,\varepsilon})\triangle(I_{i_2,\varepsilon})}\int_0^1\int_0^1 f(x,y)\phi_{i_1,i_2}(x,y)dxdy; \quad i_1,i_2 = 0(1)(m).$$

Then the following equation

$$\int_0^1\int_0^1 (f(x,y)-f_{m,\varepsilon}(x,y))^2dxdy, \tag{15}$$

achieves its minimum value and also we have

$$\int_0^1\int_0^1 f^2(x,y)dxdy = \sum_{i_1=0}^\infty\sum_{i_2=0}^\infty f^2_{i_1,i_2}\|\phi_{i_1,i_2}(x,y)\|^2. \tag{16}$$

Proof. It is an immediate consequence of theorem which was proved by Jiang and Schaufelberger (1992). □

Theorem 2. *Assume f(x,y) is continuous and is differentiable over district* $[-h,1+h]\times[-h,1+h]$, *and* $f_{m,\varepsilon_i}(x,y)$; $\varepsilon_i=\frac{ih}{k}$, *for* $i=0(1)(k-1)$, *are correspondingly M2D-BFs*(ε_0) = *2D-BFs, M2D-BFs*(ε_1), …, *M2D-BFs*(ε_{k-1}) *expansions of f(x, y) based on* $(m+1)^2$ *M2D-BFs over district D and*

$$\bar{f}_{m,k}(x,y)=\frac{1}{k}\sum_{i=0}^{k-1}f_{m,\varepsilon_i}(x,y),$$

then for sufficient large m we have:

$$\|f(x,y)-\bar{f}_{m,k}(x,y)\|_\infty\lesssim\frac{1}{k}\max_{\varepsilon_i}\|f(x,y)-f_{m,\varepsilon_i}(x,y)\|_\infty.$$

Proof. We consider $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ in the district $\left[\frac{i-1}{m},\frac{i+1}{m}\right)\times\left[\frac{i-1}{m},\frac{i+1}{m}\right)$ which are approximately equal to constants n_1 and n_2, respectively, where m is so large. Also, we use page $z=n_1x+n_2y+b$ instead of $f(x,y)$ in the district $\left[\frac{i-1}{m},\frac{i+1}{m}\right)\times\left[\frac{i-1}{m},\frac{i+1}{m}\right)$. Now in the district $\left[\frac{i}{m},\frac{i}{m}+\varepsilon_1\right)\times\left[\frac{i}{m},\frac{i}{m}+\varepsilon_1\right)$ we have:

$$\begin{aligned}\bar{f}_{m,k}(x,y)&=\frac{1}{k}\sum_{l=0}^{k-1}f_{m,\varepsilon_l}=\frac{1}{k}\sum_{l=0}^{k-1}\frac{1}{4}\left\{n_1\left(\frac{i}{m}-\frac{lh}{k}\right)+n_2\left(\frac{i}{m}-\frac{lh}{k}\right)\right.\\&\quad+b+n_1\left(\frac{i}{m}-\frac{lh}{k}\right)+n_2\left(\frac{i+1}{m}-\frac{lh}{k}\right)\\&\quad+b+n_1\left(\frac{i+1}{m}-\frac{lh}{k}\right)+n_2\left(\frac{i}{m}-\frac{lh}{k}\right)\\&\quad\left.+b+n_1\left(\frac{i+1}{m}-\frac{lh}{k}\right)+n_2\left(\frac{i+1}{m}-\frac{lh}{k}\right)+b\right\}\\&=(n_1+n_2)\left(\frac{\frac{i}{m}+\frac{i+1}{m}}{2}\right)+b-\frac{(n_1+n_2)h(k-1)}{2k},\end{aligned}\tag{17}$$

but $\frac{i+1}{m}=\frac{i}{m}+h$ and Eq. (17) can be reformulated as:

$$\bar{f}_{m,k}(x,y)=(n_1+n_2)\frac{i}{m}+b+\frac{(n_1+n_2)h}{2k}.\tag{18}$$

In other words:

$$\begin{aligned}&\max_{x,y\in\left[\frac{i}{m},\frac{i}{m}+\varepsilon_1\right)}|f(x,y)-\bar{f}_{m,k}(x,y)|\\&\simeq\max_{x,y\in\left[\frac{i}{m},\frac{i}{m}+\varepsilon_1\right)}|n_1x+n_2y+b-\bar{f}_{m,k}(x,y)|\lesssim\left|n_1\frac{i}{m}+n_2\frac{i}{m}\right.\\&\quad\left.+b-\bar{f}_{m,k}(x,y)\right|\\&=\frac{(n_1+n_2)h}{2k},\end{aligned}\tag{19}$$

so, we have:

$$\begin{aligned}&\max_{\substack{\varepsilon_i\\(x,y)\in D}}\|f(x,y)-f_{m,\varepsilon_i}(x,y)\|_\infty\geqslant\max_{\substack{\varepsilon_i\\(x,y)\in D'}}|f(x,y)\\&\quad-f_{m,\varepsilon_i}(x,y)|\simeq\left|n_1\frac{i}{m}+n_2\frac{i}{m}+b\right.\\&\quad-\frac{1}{4}\left\{n_1\frac{i}{m}+n_2\frac{i}{m}+b+n_1\frac{i}{m}+n_2\left(\frac{i}{m}+h\right)\right.\\&\quad+b+n_1\left(\frac{i}{m}+h\right)+n_2\frac{i}{m}+b+n_1\left(\frac{i}{m}+h\right)\\&\quad\left.\left.+n_2\left(\frac{i}{m}+h\right)+b\right\}\right|=\frac{(n_1+n_2)h}{2},\end{aligned}\tag{20}$$

where $D'=\left[\frac{i}{m},\frac{i}{m}+h\right)\times\left[\frac{i}{m},\frac{i}{m}+h\right)$.

By using Eqs. (19) and (20) the proof is completed. □

Theorem 3. *Let the representation error between* $f(x,y)$ *and its two-dimensional block-pulse functions,* $f_m(x,y)=f_{m,\varepsilon_0}(x,y)$ $(M2D\text{-}BFs(\varepsilon_0)=2D-BFs)$, *over the district D, as follows:*

$$e(x,y)=f(x,y)-f_m(x,y).$$

Then $\|e(x,y)\|=O(\frac{1}{m})$ *and*

$$\lim_{m\to+\infty}f_m(x,y)=\lim_{m\to+\infty}f_{m,\varepsilon_0}(x,y)=f(x,y).$$

Proof. See (Maleknejad et al., 2010). □

Theorems 2 and 3 conclude that error estimation for M2D-BFs is $\|e(x,y)\|=O\left(\frac{1}{km}\right)$.

If we assume E_1 and E_2 are errors between $f(x,y)$ and its 2D-BFs and M2D-BFs expansions, respectively, from Theorem 2 we have $E_2\leqslant\frac{1}{k}E_1$, and from (Maleknejad et al., 2010) we have $E_1\leqslant\frac{\sqrt{2}M}{m}$, where M is bounded of $\|Df(x,y)\|$ and m shows number of 2D-BFs.

So, we have

$$E_2=\|e(x,y)\|\leqslant\frac{\sqrt{2}M}{km},\tag{21}$$

where k is times of modifications of the M2D-BFs series.

Assume now that $f(x,y)$ is approximated by

$$f_{m,\varepsilon_i}(x,y)=\sum_{i_1=0}^{m}\sum_{i_2=0}^{m}f_{i_1,i_2}\phi_{i_1,i_2}(x,y),$$

whereas, $\bar{f}_{i_1,i_2}$ are the approximation of f_{i_1,i_2} and

$$\bar{f}_{m,\varepsilon_i}(x,y)=\sum_{i_1=0}^{m}\sum_{i_2=0}^{m}\bar{f}_{i_1,i_2}\phi_{i_1,i_2}(x,y),$$

then for $(x,y)\in D_{i_1,i_2}$ we have

$$\begin{aligned}\|\bar{f}_{i_1,i_2}\phi_{i_1,i_2}(x,y)-f(x,y)\|&=\|\bar{f}_{i_1,i_2}\phi_{i_1,i_2}(x,y)-f(x,y)\\&\quad-f_{i_1,i_2}\phi_{i_1,i_2}(x,y)\\&\quad+f_{i_1,i_2}\phi_{i_1,i_2}(x,y)\|\\&\leqslant\|f_{i_1,i_2}\phi_{i_1,i_2}(x,y)-f(x,y)\|\\&\quad+\|\bar{f}_{i_1,i_2}\phi_{i_1,i_2}(x,y)\\&\quad-f_{i_1,i_2}\phi_{i_1,i_2}(x,y)\|.\end{aligned}\tag{22}$$

We have

$$\begin{aligned}&\|\bar{f}_{i_1,i_2}\phi_{i_1,i_2}(x,y)-f_{i_1,i_2}\phi_{i_1,i_2}(x,y)\|\\&\quad=\left(\int_{I_{i_1,\varepsilon_i}}\int_{I_{i_2,\varepsilon_i}}(\bar{f}_{i_1,i_2}\phi_{i_1,i_2}(x,y)-f_{i_1,i_2}\phi_{i_1,i_2}(x,y))^2dydx\right)^{\frac{1}{2}}\\&\quad=|\bar{f}_{i_1,i_2}-f_{i_1,i_2}|\left(\int_{I_{i_1,\varepsilon_i}}\int_{I_{i_2,\varepsilon_i}}dydx\right)^{\frac{1}{2}}\\&\quad=\triangle(I_{i_1,\varepsilon_i})\triangle(I_{i_2,\varepsilon_i})|\bar{f}_{i_1,i_2}-f_{i_1,i_2}|\\&\quad\leqslant\triangle(I_{i_1,\varepsilon_i})\triangle(I_{i_2,\varepsilon_i})\|\bar{f}_m-f\|_\infty.\end{aligned}\tag{23}$$

Consequently by using Eqs. (21)–(23), the following error bound is obtained:

$$\|\bar{f}_{i_1,i_2}\phi_{i_1,i_2}-f(x,y)\|\leqslant\frac{\sqrt{2}M}{km}+\triangle(I_{i_1,\varepsilon_i})\triangle(I_{i_2,\varepsilon_i})\|\bar{f}_m-f\|_\infty.\tag{24}$$

Moreover Eq. (24) implies that:

$$\lim_{m\to+\infty}f_{m,\varepsilon_i}(x,y)=f(x,y).\tag{25}$$

4. Method of solution

In this section, we solve mixed nonlinear Volterra–Fredholm type integral equations of the first kind of the form Eq. (1) with Eq. (2) by using M2D-BFs.

We now approximate functions $u(x,y)$,$f(x,y)$,$[u(x,y)]^p$ and $k(x,y,s,t)$ with respect to M2D-BFs by manipulation as Section 2:

$$\begin{cases} u(x,y) \simeq \Phi^T_{m,\varepsilon}(x,y)U_{m,\varepsilon}, \\ f(x,y) \simeq \Phi^T_{m,\varepsilon}(x,y)F_{m,\varepsilon}, \\ (u(x,y))^p \simeq \Phi^T_{m,\varepsilon}(x,y)U_{m,\varepsilon,p}, \\ k(x,y,s,t) \simeq \Phi^T_{m,\varepsilon}(x,y)K_{m,\varepsilon}\Phi_{m,\varepsilon}(s,t), \end{cases} \tag{26}$$

where $\Phi_{m,\varepsilon}(x,y)$ is defined in Eq. (8), the vectors $U_{m,\varepsilon}$, $F_{m,\varepsilon}$, $U_{m,\varepsilon,p}$, and matrix $K_{m,\varepsilon}$ are M2D-BFs coefficients of $u(x,y)$,$f(x,y)$, $[u(x,y)]^p$ and $k(x,y,s,t)$ respectively.

Lemma 1. *Let* $(m + 1)^2$*-vectors* $U_{m,\varepsilon}$ *and* $U_{m,\varepsilon,p}$ *be M2D-BFs coefficients of* $u(x,y)$ *and* $[u(x,y)]^p$*, respectively. If*

$$U_{m,\varepsilon} = [u_{0,0},\ldots,u_{0,m},\ldots,u_{m,0},\ldots,u_{m,m}]^T, \tag{27}$$

then we have:

$$U_{m,\varepsilon,p} = \left[u^p_{0,0},\ldots,u^p_{0,m},\ldots,u^p_{m,0},\ldots,u^p_{m,m}\right]^T, \tag{28}$$

where $p \geqslant 1$, is a positive integer.

Proof. (By induction) When $p = 1$, Eq. (28) follows at once from $[u(x,y)]^p = u(x,y)$. Suppose that Eq. (28) holds for p, we shall deduce it for $(p + 1)$. Since $[u(x,y)]^{p+1} = u(x,y)[u(x,y)]^p$, from Eqs. (26) and (10) it follows that

$$\begin{aligned}[u(x,y)]^{p+1} &= u(x,y)[u(x,y)]^p \\ &= U^T_{m,\varepsilon}\Phi_{m,\varepsilon}(x,y)\Phi^T_{m,\varepsilon}(x,y)U_{m,\varepsilon,p} \\ &= U^T_{m,\varepsilon}\widetilde{U}_{m,\varepsilon,p}\Phi_{m,\varepsilon}(x,y).\end{aligned} \tag{29}$$

Now by using Eq. (28) we obtain

$$U^T_{m,\varepsilon}\widetilde{U}_{m,\varepsilon,p} = \left[u^{p+1}_{0,0},\ldots,u^{p+1}_{0,m},\ldots,u^{p+1}_{m,0},\ldots,u^{p+1}_{m,m}\right]^T, \tag{30}$$

therefore Eq. (28) holds for $(p + 1)$, and the lemma is established. □

To approximate the integral part in Eq. (1) with Eq. (2), from Eq. (26) we get

$$\begin{aligned}&\int_0^x\int_0^1 k(x,y,s,t)[u(s,t)]^p dtds \\ &\simeq \int_0^x\int_0^1 \Phi^T_{m,\varepsilon}(x,y)K_{m,\varepsilon}\Phi_{m,\varepsilon}(s,t)\Phi^T_{m,\varepsilon}(s,t)U_{m,\varepsilon,p}dtds \\ &= \Phi^T_{m,\varepsilon}(x,y)K_{m,\varepsilon}\left(\int_0^x\int_0^1 \Phi_{m,\varepsilon}(s,t)\Phi^T_{m,\varepsilon}(s,t)dtds\right)U_{m,\varepsilon,p}.\end{aligned} \tag{31}$$

Now by using Eqs. (5) and (9), denoting R_j for the $(j + 1)$th row of the conventional integration operational matrix $P_{m,\varepsilon}$ ($(P_{m,\varepsilon})_{(m+1)\times(m+1)}$ is operational matrix of 1D-BFs defined over [0,1), see Maleknejad and Mahdiani, 2011) and considering $\int_0^1\phi_i(t)dt = \triangle(I_{i,\varepsilon})$ follows:

$$\begin{aligned}&\int_0^x\int_0^1 \Phi_{m,\varepsilon}(s,t)\Phi^T_{m,\varepsilon}(s,t)dtds \\ &= \begin{pmatrix} \int_0^x\int_0^1\phi_0(s)\phi_0(t)dtds & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \int_0^x\int_0^1\phi_0(s)\phi_m(t)dtds & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \int_0^x\int_0^1\phi_m(s)\phi_m(t)dtds \end{pmatrix}_{(m+1)^2\times(m+1)^2} \\ &= \begin{pmatrix} (h-\varepsilon)R_0\Phi_{m,\varepsilon}(x) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & hR_0\Phi_{m,\varepsilon}(x) & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon R_0\Phi_{m,\varepsilon}(x) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & (h-\varepsilon)R_m\Phi_{m,\varepsilon}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & \varepsilon R_m\Phi_{m,\varepsilon}(x) \end{pmatrix}_{(m+1)^2\times(m+1)^2}.\end{aligned} \tag{32}$$

Also by using Eq. (5), Eq. (8) can be reformulated as:

$$\Phi_{m,\varepsilon}(x,y) = \begin{pmatrix} \phi_0(x) & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \phi_0(x) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \phi_m(x) & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & \phi_m(x) \end{pmatrix}_{(m+1)^2\times(m+1)^2} . [\phi_0(y),\dots,\phi_m(y),\dots,\phi_0(y),\dots,\phi_m(y)]^T_{(m+1)^2\times 1}. \quad (33)$$

So, we have

$$\Phi^T_{m,\varepsilon}(x,y)K_{m,\varepsilon} = [\phi_0(y),\dots,\phi_m(y),\dots,\phi_0(y),\dots,\phi_m(y)]_{(m+1)^2\times 1} \times \begin{pmatrix} k_{1,1}\phi_0(x) & \dots & k_{1,(m+1)}\phi_0(x) & \dots & k_{1,m(m+1)}\phi_0(x) & \dots & k_{1,(m+1)^2}\phi_0(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{(m+1),1}\phi_0(x) & \dots & k_{(m+1),(m+1)}\phi_0(x) & \dots & k_{(m+1),m(m+1)}\phi_0(x) & \dots & k_{(m+1),(m+1)^2}\phi_0(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{m(m+1),1}\phi_m(x) & \dots & k_{m(m+1),(m+1)}\phi_m(x) & \dots & k_{m(m+1),m(m+1)}\phi_m(x) & \dots & k_{m(m+1),(m+1)^2}\phi_m(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{(m+1)^2,1}\phi_m(x) & \dots & k_{(m+1)^2,(m+1)}\phi_m(x) & \dots & k_{(m+1)^2,m(m+1)}\phi_m(x) & \dots & k_{(m+1)^2,(m+1)^2}\phi_m(x) \end{pmatrix}_{(m+1)^2\times(m+1)^2}. \quad (34)$$

Also, we have:

$$R_i\Phi(x) = \begin{cases} \frac{(h-\varepsilon)}{2}\phi_0(x) + (h-\varepsilon)\phi_1(x) + \dots + (h-\varepsilon)\phi_m(x), & i=0 \\ \frac{h}{2}\phi_i(x) + h\phi_{i+1}(x) + \dots + h\phi_m(x), & i = 1(1)(m-1), \\ \frac{\varepsilon}{2}\phi_m(x), & i=m \end{cases}$$

and

$$\phi_i(x)\phi_j(x) = \begin{cases} \phi_i(x), & i=j \\ 0, & \text{otherwise} \end{cases}. \quad (35)$$

By using Eqs. (32), (34) and (35), Eq. (31) can be reformulated as:

$$[\phi_0(y),\dots,\phi_m(y),\dots,\phi_0(y),\dots,\phi_m(y)]_{(m+1)^2\times 1} . \begin{pmatrix} A_{00} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ A_{10} & A_{11} & \mathbf{0} & \dots & \mathbf{0} \\ A_{20} & A_{21} & A_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m0} & A_{m1} & A_{m2} & \dots & A_{mm} \end{pmatrix}_{(m+1)^2\times(m+1)^2} .U_{m,\varepsilon,p}, \quad (36)$$

where

$$A_{i,j} = \begin{cases} \frac{\triangle(I_{j,\varepsilon})}{2}\triangle(I_{r,\varepsilon})k_{lz}\phi_i(x), & i=j \\ & , \\ \triangle(I_{j,\varepsilon})\triangle(I_{r,\varepsilon})k_{lz}\phi_i(x), & \textit{otherwise} \end{cases} \quad (37)$$

where

$$l = ((m+1)i+1)(1)((m+1)(i+1)),$$
$$z = ((m+1)j+1)(1)((m+1)(j+1)),$$
$$r = z - (m+1)\left[\frac{z}{(m+1)}\right],$$

and $\mathbf{0}$ is a zero matrix. Also

$$\begin{pmatrix} A_{00} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ A_{10} & A_{11} & \mathbf{0} & \dots & \mathbf{0} \\ A_{20} & A_{21} & A_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m0} & A_{m1} & A_{m2} & \dots & A_{mm} \end{pmatrix}_{(m+1)^2\times(m+1)^2} = \begin{pmatrix} \phi_0(x) & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \phi_0(x) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \phi_m(x) & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & \phi_m(x) \end{pmatrix}_{(m+1)^2\times(m+1)^2} .Q, \quad (38)$$

where

$$Q = \begin{pmatrix} Q_{00} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ Q_{10} & Q_{11} & \mathbf{0} & \dots & \mathbf{0} \\ Q_{20} & Q_{21} & Q_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{m0} & Q_{m1} & Q_{m2} & \dots & Q_{mm} \end{pmatrix}_{(m+1)^2\times(m+1)^2}, \quad (39)$$

Table 1 Numerical results of Example 1 with M2D-BFs.

Nodes (x,y)	Error for $m=8$		
$(x,y)=2^{-l}$	$k=1$	$k=2$	$k=3$
$l=1$	0.03748936	0.02837304	0.02522153
$l=2$	0.05090571	0.03943282	0.03190341
$l=3$	0.02574872	0.01813532	0.01391462
$l=4$	0.04112879	0.02553757	0.02222506

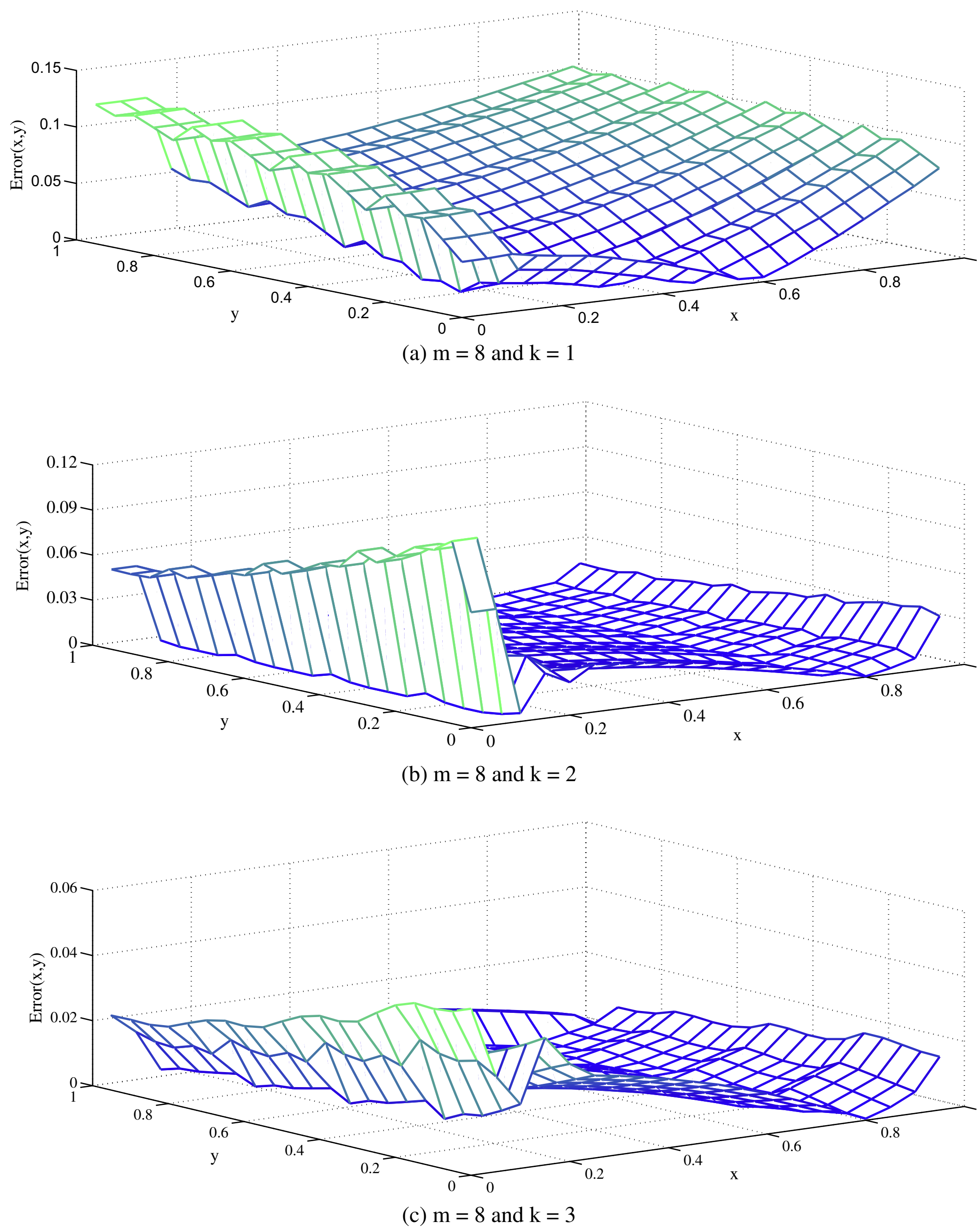
Table 2 Error results for Example 1.

Nodes (x,y)	Present method	Method of Maleknejad and Mahdiani (2011)
$(x,y)=2^{-l}$	$m=8$ and $k=2$	$m=16$
$l=1$	0.02837304	0.0288649
$l=2$	0.03943282	0.0398778
$l=3$	0.01813532	0.0310669
$l=4$	0.02553757	0.0277814

$$Q_{i,j}=\begin{cases}\frac{\triangle(I_{j,\varepsilon})}{2}\triangle(I_{r,\varepsilon})k_{lz}, & i=j\\ \triangle(I_{j,\varepsilon})\triangle(I_{r,\varepsilon})k_{lz}, & \text{otherwise}\end{cases}. \tag{40}$$

So, we have :

$$\int_0^x\int_0^1 k(x,y,s,y)[u(s,t)]^p dtds \simeq \Phi^T_{m,\varepsilon}(x,y)QU_{m,\varepsilon,p}. \tag{41}$$



(a) m = 8 and k = 1

(b) m = 8 and k = 2

(c) m = 8 and k = 3

Figure 1 Absolute value of error, Example 1 with $m=8$ and $k=1,2,3$.

Table 3 Numerical results of Example 2 with M2D-BFs.

Nodes (x, y)	Error for $m = 8$		
$(x, y) = 2^{-l}$	$k = 1$	$k = 2$	$k = 3$
$l = 1$	0.04289052	0.03148406	0.02743874
$l = 2$	0.06470576	0.04585369	0.04071913
$l = 3$	0.03803418	0.02437943	0.02249845
$l = 4$	0.03592263	0.02836157	0.02045372

Substituting Eqs. (26) and (41) into Eq. (1) with Eq. (2) gives:

$$\Phi^T_{m,\varepsilon}(x, y)F_{m,\varepsilon} = \Phi^T_{m,\varepsilon}(x, y)QU_{m,\varepsilon,p} \Rightarrow F_{m,\varepsilon} = QU_{m,\varepsilon,p}. \quad (42)$$

After solving the above nonlinear system by using Newton–Raphson method, we can find $U_{m,\varepsilon}$ and then

$$u_{m,\varepsilon}(x, y) = U^T_{m,\varepsilon}\Phi_{m,\varepsilon}(x, y). \quad (43)$$

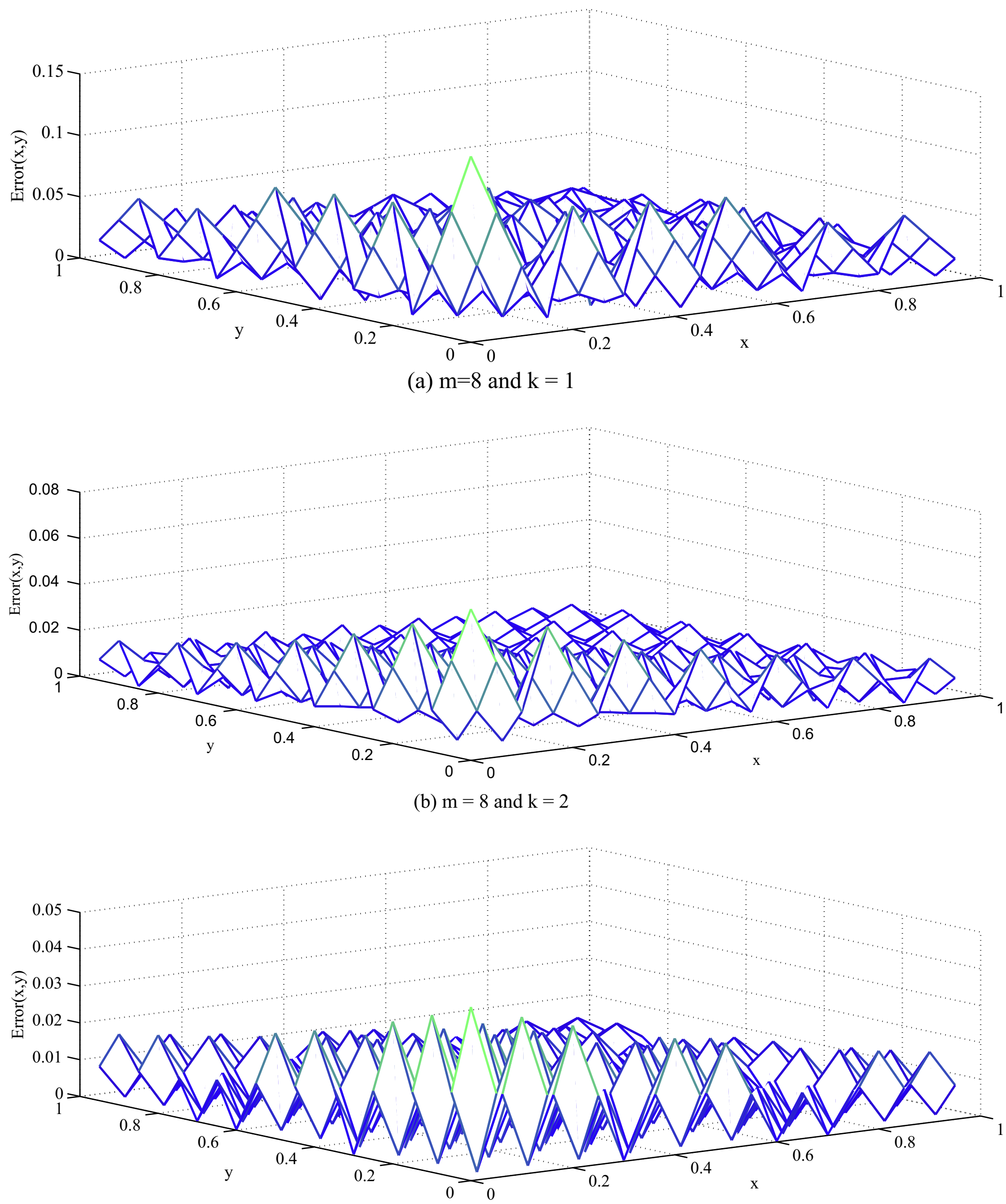
Then

$$u(x, y) \simeq \bar{u}_{m,k}(x, y) = \frac{1}{k}\sum_{i=0}^{k-1} u_{m,\varepsilon_i}(x, y), \quad (44)$$

where $\varepsilon_i = \frac{ih}{k}$, $i = 0(1)(k-1)$ is the estimation of the solution of mixed nonlinear Volterra–Fredholm type integral equation of the first kind.

5. Numerical examples

In this section to demonstrate the effectiveness of our approach several examples are presented. All results are computed by using a program written in the Matlab. The



(a) m=8 and k = 1

(b) m = 8 and k = 2

(c) m = 8 and k = 3

Figure 2 Absolute value of error, Example 2 with $m = 8$ and $k = 1,2,3$.

Table 4 Error results for Example 2.

Nodes (x,y)	Present method	Method of Maleknejad and Mahdiani (2011)
$(x,y)=2^{-l}$	$m=8$ and $k=2$	$m=16$
$l=1$	0.03148406	0.04289006
$l=2$	0.04585369	0.04589757
$l=3$	0.02437943	0.04034072
$l=4$	0.02836157	0.04312157

numerical experiments are carried our for the selected grid point which are proposed as $(2^{-l};\ l=1,2,3,4)$ and m terms and k times of modifications of the M2D-BFs series. The following problems have been tested.

Example 1. Consider the following mixed linear Volterra–Fredholm type integral equation (Maleknejad and Mahdiani, 2011):

$$\int_0^x \int_0^1 \cos(y-t)e^{s-x}u(s,t)dtds = f(x,y); \quad (x,y) \in [0,1) \times \Omega, \tag{45}$$

where

$$f(x,y) = \frac{1}{4}xe^{-x}(2\cos(y) + \sin(2-y) + \sin(y)). \tag{46}$$

The exact solution is $u(x,y) = e^{-x}\cos(y)$. Table 1 and Fig. 1 illustrate the numerical results for this example.

The error results for proposed method besides the error for method of Maleknejad and Mahdiani (2011) are tabulated in Table 2.

Example 2. Consider the following mixed nonlinear Volterra–Fredholm type integral equation (Maleknejad and Mahdiani, 2011):

$$\int_0^x \int_0^1 (t+y)e^{2s-x}u^2(s,t)dtds = f(x,y); \quad (x,y) \in [0,1) \times \Omega, \tag{47}$$

where

$$f(x,y) = \frac{1}{2}xye^{-x} + \frac{1}{4}xe^{-x} - \frac{1}{2}xye^{-x-2} - \frac{3}{4}xe^{-x-2}. \tag{48}$$

The exact solution is $u(x,y) = e^{-x-y}$. Table 3 and Fig. 2 illustrate the numerical results for this example.

The error results for proposed method besides the error for method of Maleknejad and Mahdiani (2011) are tabulated in Table 4.

6. Conclusion

In this paper a computational method for approximate solution of mixed nonlinear Volterra–Fredholm type integral equations of the first kind, based on the expansion of the solution as series of M2D-BFs was presented. This method converts a mixed nonlinear Volterra–Fredholm type integral equation whose answer is the coefficients of M2D-BFs expansion of the solution of mixed nonlinear Volterra–Fredholm type integral equation. Also, we have shown that our approach is convergent and its order of convergence is $O\left(\frac{1}{km}\right)$. This method can be easily extended and applied to mixed nonlinear Volterra–Fredholm type integral equations of the second kind and nonlinear system of the mixed Volterra–Fredholm type integral equations.

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