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طريقة اضطراب هوموتوبي المعدلة لحل نظام من المعادلات الخطية

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الملخص:

لقد تم في هذة الدراسة استخدام طريقة اضطراب هوموتوبي المعدلة لحل نظام من المعادلات الخطية. لقد بينت الدراسة بأن هذه الطريقة تمكننا من ايجاد الحل المؤكد لنظام من المعادلات الخطية. كما إن الطريقة المستخدمة في هذه الدراسة لاتعتمد على المتغير الوسيط المساعد والمؤثر المساعد والنتائج التي تعطيها يمكن النظر إليها بأنها تحسين أصيل وامتداد للنتائج المعروفة سابقا.

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ORIGINAL ARTICLE

Modified homotopy perturbation method for solving system of linear equations

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Abstract In this paper, we use the modified homotopy perturbation method to solving the system of linear equations. We show that this technique enables us to find the exact solution of the system of linear equations. This technique is independent of the auxiliary parameter and auxiliary operator. Our results can be viewed as a novel improvement and an extension of the previously known results.

1. Introduction

Consider the system of linear equations

$$AU = b, \tag{1.1}$$

where

$$A = [a_{ij}], \quad U = [u_j] \text{ and } b = [b_j], \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n.$$

It is well-known that a wide class of problems, which arise in several branches of pure and applied sciences can be studied in the unified and general framework of the system of linear Eq. (1.1), see Burden and Faires (2001). Several methods and techniques have been developed for solving system of linear Eq. (1.1). Liu (2011) and Karamati (2009) used homotopy perturbation method to suggest some iterative methods for solving a system of linear equations. We would like to mention that the Homotopy perturbation method was developed by He (1999) and has been used for solving a wide class of problems arising in various branches of pure and applied sciences. It has been shown that homotopy perturbation method is very reliable and efficient. Noor (2010a, 2010b) introduced a modified homotopy perturbation method by combining elegantly the homotopy analysis method Liao (2004) and homotopy perturbation technique of He (1999, 2000, 2003, 2004, 2005). Noor (2010a, 2010b) used this modified homotopy perturbation technique to develop various iterative methods for solving nonlinear equations. In this paper, we again use this modified homotopy perturbation technique for solving system of linear Eq. (1.1). It has been shown that the modified homotopy perturbation method provides us the exact solution of the system of linear equations as compared to the series of solution obtained by using homotopy perturbation in Liu (2011). Yusufoglu (2009) has also considered the similar technique with different auxiliary parameter and has obtained the exact solution of the linear system of equations. Our technique is more general and flexible than the technique of Yusufoglu (2009). Our technique is quite effective in finding the exact

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solutions of the system of linear equations. It is an interesting problem to consider this technique for finding the exact solutions of the system of nonlinear equations and its variant forms.

2. Modified homotopy perturbation method

To convey the basic idea of the modified homotopy perturbation method, we assume that

$$L(U) = AU - b, \tag{2.1}$$

and for any splitting matrix Q

$$F(U) = QU - W_0. \tag{2.2}$$

We define a new homotopy $H(U,p,T): R^n \times [0,1] \times R^n \to R^n$ as:

$$H(U,p,T) = (1-p)F(U) + (qH)pL(U) - p^2(1-p)T = 0, \tag{2.3}$$

where $p \in [0,1]$ is an embedding parameter, $q \neq 0$ an auxiliary parameter, H is the auxiliary matrix, W_0 is the initial approximation and $T \in R^n$ is an arbitrary operator. This modified hompotopy perturbation method is mainly due to Noor (2010a,.

By using (2.1) and (2.2) in (2.3), we have

$$\begin{aligned} H(U,p,T) = QU - W_0 - p(QU - W_0) + (qH)p(AU - b) \\ - p^2(1-p)T = 0, \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), it is clear that

$$H(U,0,T) = F(U) = QU - W_0 = 0. \tag{2.5}$$

$$H(U,1,T) = L(U) = AU - b = 0. \tag{2.6}$$

The embedding parameter p increases monotonically from zero to unity as trivial problem $H(U,0,T) = F(U)$ is continuously deformed to original problem $H(U,1,T) = L(U)$. The changing process of p from zero to unity is called deformation. $H(U,0,T) = F(U)$ and $H(U,1,T) = L(U)$ are holomorphic. The basic assumption is that the solution of (2.3) can be expressed as a power series in p:

$$U = U_0 + U_1p + U_2p^2 + \cdots. \tag{2.7}$$

The approximate solution of (1.1) can be obtained as

$$V = \lim_{p\to 1} U = U_0 + U_1 + U_2 + \cdots = \sum_{k=1}^{\infty} U_k. \tag{2.8}$$

By substituting (2.7) in (2.4), we have

$$\begin{aligned} &Q[U_0 + U_1p + U_2p^2 + \cdots] - W_0 \\ &\quad - p(Q[U_0 + U_1p + U_2p^2 + \cdots] - W_0) \\ &\quad + (qH)p(A[U_0 + U_1p + U_2p^2 + \cdots] - b) - p^2(1-p)T = 0. \end{aligned} \tag{2.9}$$

Equating the coefficients of identical powers of p, we get

$$\begin{cases} p^0 : QU_0 - W_0 = 0, \\ p^1 : QU_1 + (qHA - Q)U_0 + W_0 - qHb = 0, \\ p^2 : QU_2 + (qHA - Q)U_1 - T = 0, \\ p^3 : QU_3 + (qHA - Q)U_2 + T = 0, \\ p^k : QU_k + (qHA - Q)U_{k-1} = 0, k = 4,5,\cdots. \end{cases} \tag{2.10}$$

Thus, from (2.10) we have the following iterative scheme:

$$\begin{cases} U_0 = Q^{-1}W_0, \\ U_1 = (I - qQ^{-1}HA)U_0 + Q^{-1}(qHb - W_0), \\ U_2 = (I - qQ^{-1}HA)U_1 + Q^{-1}T, \\ U_3 = (I - qQ^{-1}HA)U_2 - Q^{-1}T, \\ U_k = (I - qQ^{-1}HA)U_{k-1}, \quad k = 4,5,\cdots. \end{cases} \tag{2.11}$$

Taking the initial approximation $W_0 = qHb$, we have

$$\begin{cases} U_0 = q(Q^{-1}H)b, \\ U_1 = (I - qQ^{-1}HA)U_0, \\ U_2 = (I - qQ^{-1}HA)U_1 + Q^{-1}T, \\ U_3 = (I - qQ^{-1}HA)U_2 - Q^{-1}T, \\ U_k = (I - qQ^{-1}HA)U_{k-1}, \quad k = 4,5,\cdots. \end{cases} \tag{2.12}$$

To find the operator T, we may take either $U_2 = 0$ or $U_3 = 0$. We remark that, if $U_2 = 0$, then we obtain the same series of solution derived by Liu (2011). Here, we consider the case $U_3 = 0$. From (2.12), we have

$$U_3 = (I - qQ^{-1}HA)U_2 - Q^{-1}T = \mathbf{0}. \tag{2.13}$$

Using the value of U_2 from (2.12) in (2.13), we have

$$(I - qQ^{-1}HA)[(I - qQ^{-1}HA)U_1 + Q^{-1}T] - Q^{-1}T = 0.$$

From which, we have

$$Q^{-1}T = [qQ^{-1}HA]^{-1}(I - qQ^{-1}HA)^2U_1. \tag{2.14}$$

Thus by using (2.14) in (2.12), we obtain the new iterative scheme

$$\begin{cases} U_0 = q(Q^{-1}H)b, \\ U_1 = (I - qQ^{-1}HA)U_0 = (I - qQ^{-1}HA)q(Q^{-1}H)b, \\ U_2 = ([qQ^{-1}HA]^{-1} - I)(I - qQ^{-1}HA)q(Q^{-1}H)b, \\ U_k = \mathbf{0}, \quad k = 3,4,\ldots. \end{cases} \tag{2.15}$$

Thus, the solution U is obtained as:

$$\begin{aligned} U &= U_0 + U_1p + U_2p^2 + \cdots, \\ &= q(Q^{-1}H)b + p(I - qQ^{-1}HA)q(Q^{-1}H)b \\ &\quad + p^2([qQ^{-1}HA]^{-1} - I)(I - qQ^{-1}HA)q(Q^{-1}H)b + 0 + 0\cdots. \end{aligned} \tag{2.16}$$

Hence by setting $p = 1$, we have the following solution V as in (2.8):

$$\begin{aligned} V &= \sum_{k=1}^{\infty} U_k \\ &= q(Q^{-1}H)b + (I - qQ^{-1}HA)q(Q^{-1}H)b \\ &\quad + ([qQ^{-1}HA]^{-1} - I)(I - qQ^{-1}HA)q(Q^{-1}H)b. \end{aligned} \tag{2.17}$$

Thus, by simplifying (2.17), we obtain

$$V = [qQ^{-1}HA]^{-1}q(Q^{-1}H)b = A^{-1}b, \tag{2.18}$$

which is the exact solution of (1.1).

Remark 2.1. For the convergence analysis of the modified homotopy perturbation method, see He (2003, 2004).

3. Conclusion

In this paper, we have used the modified homotopy perturbation technique to find the exact solution of the system of linear equations. It has been shown that with suitable modification of the homotopy perturbation technique, one can find the exact solutions of the nonlinear equations. This technique may stimulate further research.

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