The non-linear dynamics of a mass resting on an inverted equal-angle wedge was analyzed using both numerical and semi-numerical methods. The energy balance method (EBM) and the homotopy perturbation method (HOM) were used to formulate the capacity and frequency. The governing equation was solved by the differential transformation method (DTM) in the semi-numerical approach. The effects of intermediate variables were evaluated. Comparison results with exact and numerical solutions were considered to evaluate and compare the performance of each method. The performance and capability of each method have been revealed and discussed.

A. Mirzabeigy et al.
Nonlinear dynamics of a particle on a rotating parabola via the analytic and semi-analytic approaches

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Abstract In present study, nonlinear dynamics of a particle on a rotating parabola are analyzed by means of the analytic and semi-analytic approaches. The Energy balance method (EBM), homotopy perturbation method (HPM) and amplitude–frequency formulation (AFF) are applied as the analytic approaches and the frequency-amplitude relationships are obtained. The governing equation of motion is also solved by the differential transform method (DTM) as a semi-analytic approach. The effects of different parameters on the governing equation are evaluated. Comparison of results with exact and numerical solutions are investigated. The performance and capability of each method are revealed and discussed.

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1. Introduction

Many phenomena in applied sciences and engineering expressed as nonlinear differential equations. This issue especially in mechanics and physics for dynamics and oscillations analysis is visible. In recent years, remarkable attention has been directed toward solutions of these nonlinear problems and researchers developed many methods. Among these methods, some of them are proposed by Prof. He and called He’s methods such as: energy balance method (He, 2002), homotopy perturbation method (He, 2009), amplitude–frequency formulation (He, 2008a), max–min approach (He, 2008b), Hamiltonian approach (He, 2008a), variational approach (He, 2007, 2011a,b; Zhou and He, 2010), Parameter expanding method (He, 2006, 2008c) and variational iteration method (He et al., 2010). These analytic methods successfully served to analysis of nonlinear problems. For example; the oscillation of a mass attached to a stretched elastic wire (Durraz et al., 2011; Xu, 2010), cantilever beam vibration with nonlinear boundary condition (Sedghi and Shirazi, 2012), nonlinear oscillations of a punctual charge in the electric field of a charged ring (Yildirim et al., 2011), analytical solution for magnetohydrodynamic flows of viscoelastic fluids in converging/diverging channels (Shadlo and Kimiaeifard, 2011) and many other problems (Askari et al., 2010; Belendez et al., 2009; Chen et al., 2011; Cvetinac, 2006; Ganji et al., 2010; Kimiaeifard et al., 2011; Mehdipour et al., 2010; Ozi and Yildirim, 2007; Xu and He, 2010; Yildirim et al., 2010, 2012a,b; Yonesian et al., 2010, 2011a,b; Zhang, 2009) are solved by carrying out the He’s methods. Besides these methods, there exist other techniques for solving nonlinear problems, that one of them is the differential transform method. The Differential transform method is a semi-analytic
method, based on Taylor expansion and does not require any linearization and small perturbation. This method has been exerted to structural dynamics (Kaya and Özgumus, 2007; Yalcin et al., 2009), heat transfer problems (Yaghoobi and Torabi, 2011) and so on (El-Shahed, 2008; Momani and Ertürk, 2008).

In this study, the nonlinear dynamics of a particle on a rotating parabola are considered. The governing equation introduced by Nayfeh and Mook (1979):

\[ u'' + 4q^2u^2u'' + c^2u + 4q^2u'^2 = 0, \quad u(0) = A, \quad u'(0) = 0. \]

(1)

We solve Eq. (1) via the energy balance method, homotopy perturbation method, amplitude–frequency formulation and differential transform method and examine the advantages and disadvantages of each method by comparison of results with exact and fourth-order Runge–Kutta solutions.

2. Solution procedure

2.1. The energy balance method (EBM)

In energy balance method, first the variational principle obtained and then the Hamiltonian is constructed, finally by collocation method, one can yield the angular frequency.

For the nonlinear equation presented in Eq. (1), the variational principle can be obtained as:

\[ J(u) = \int_0^t \left( -\frac{1}{2} \dot{u}^2(1 + 4q^2u^2) + \frac{c^2}{2} \dot{u}^2 - \frac{e^2}{2} A^2 \right) dt. \]

(2)

Its Hamiltonian and residue \( R \), therefore, can be written in the form:

\[ H = \frac{1}{2} \dot{u}^2(1 + 4q^2u^2) + \frac{c^2}{2} \dot{u}^2 - \frac{e^2}{2} A^2 \]

(3)

\[ R(t) = \frac{1}{2} \dot{u}^2(1 + 4q^2u^2) + \frac{c^2}{2} \dot{u}^2 - \frac{e^2}{2} A^2 = 0. \]

(4)

For satisfied initial condition in Eq. (1), assume the approximate solution in the form of:

\[ u(t) = A \cos \omega t. \]

(5)

Substituting Eq. (5) into Eq. (4), yield:

\[ R(t) = \frac{1}{2} \left( -\omega A \sin \omega t \right)^2(1 + 4q^2(A \cos \omega t)^2) + \frac{c^2}{2} (A \cos \omega t)^2 - \frac{e^2}{2} A^2 = 0. \]

(6)

From Eq. (6), we obtained the following result:

\[ \omega = \frac{\sqrt{2}}{A \sin \omega t} \sqrt{\frac{\frac{c^2}{2} A^2 - \omega^2 (A \cos \omega t)^2}{1 + 4q^2(A \cos \omega t)^2}}. \]

(7)

Finally collocation at \( \omega t = \frac{\pi}{2} \) gives:

\[ \omega_{\text{EBM}} = \frac{\frac{c}{A}}{\sqrt{1 + 2q^2 A^2}}. \]

(8)

2.2. The homotopy perturbation method (HPM)

Based on standard procedure of the homotopy perturbation method, we first by using Eq. (1) establish the following homotopy:

\[ u'' + 1 \cdot u = p[-c^2u - 4q^2u'' - 4q^2u'^2 + u], \quad p \in [0, 1] \]

(9)

It is obvious that when \( p = 0 \), Eq. (9) becomes a linear ordinary differential equation and when \( p = 1 \), it becomes the original nonlinear equation. We consider \( u \) and \( 1 \) as series of \( p \) in the following form:

\[ 1 = o^2 - px_1 - p^2x_2 \ldots, \]

(10)

\[ u = u_0 + pu_1 + p^2u_2 \ldots. \]

(11)

Substituting Eqs. (10) and (11) into Eq. (9) yields:

\[ (u_0 + pu_1 + p^2u_2 \ldots)'' + (\omega^2 \cdot (u_0 + pu_1 + p^2u_2 \ldots) = p[-c^2(u_0 + pu_1 + p^2u_2 \ldots) - 4q^2(u_0 + pu_1 + p^2u_2 \ldots)'' - 4q^2(u_0 + pu_1 + p^2u_2 \ldots)'' + (u_0 + pu_1 + p^2u_2 \ldots)]. \]

(12)

By expanding Eq. (12) and collecting terms with same power, we can find two first linear equations with initial conditions as follows:

\[ p^0: u_0'' + \omega^2 u_0 = 0, \quad u_0(0) = A, u_0'(0) = 0. \]

(13)

\[ p^1: u_1'' + \omega^2 u_1 = -c^2u_0 - 4q^2u_0'' - 4q^2u_0u_0'' + u_0(1 + \kappa). \]

(14)

Solving Eq. (13) gives:

\[ u_0(t) = A \cos \omega t. \]

(15)

Substituting Eq. (15) into Eq. (14) yields:

\[ u_1'' + \omega^2 u_1 = -c^2A \cos \omega t - 4q^2(A \cos \omega t)^3(-\omega^2A \cos \omega t) - 4q^2(A \cos \omega t)(-\omega A \sin \omega t)^2 + (A \cos \omega t) \]

\[ \times (1 + \kappa). \]

(16)

Avoiding secular term in \( u_1 \), requires:

\[ \int_0^\pi \left[ -c^2A \cos \omega t - 4q^2A^2 \cos \omega t - 4q^2A^2 \cos \omega t \cos \omega t \sin^2 \omega t \]

\[ + A(1 + \kappa) \cos \omega t \cos \omega t \right] dt = 0. \]

(17)

From Eq. (17) we obtain:

\[ \kappa = c^2 - 1 - 4q^2A^2 \omega^2. \]

(18)

Setting \( p = 1 \) in Eq. (10), we have:

\[ \omega^2 = 1 + \kappa. \]

(19)

First-order approximate solution can be obtained by substituting Eq. (18) into Eq. (19) as:

\[ \omega_{\text{HPM}} = \frac{c}{\sqrt{1 + 2q^2A^2}}. \]

(20)

2.3. The amplitude–frequency formulation (AFF)

Based on standard procedure of the amplitude–frequency formulation, we consider two trial functions \( u_1(t) = A \cos \omega t \) and \( u_2(t) = A \cos \omega t \), respectively. Which are the solutions of the following linear equations:

\[ \ddot{u} + \omega^2 u = 0, \quad \omega^2 = 1 \]

(21)

\[ \ddot{u} + \omega_2^2 u = 0, \quad \omega_2^2 = \omega^2. \]

(22)
Substituting the mentioned trial functions into Eq. (1), results in the following residuals:

\begin{align*}
R_1(t) &= -A \cos t - 4q^2 A^3 \cos^3 t + \varepsilon^2 A \cos t + 4q^2 A^3 \cos t \sin^2 t, \\
R_2(t) &= -A \cos t - 4q^2 A^3 \cos^3 t + \varepsilon^2 A \cos t + 4q^2 A^3 \cos t \sin^2 t.
\end{align*}

(23)

According to the AFF, the residuals rewritten in the form of weighted residuals as follows:

\begin{align*}
\tilde{R}_1 &= \frac{4}{T_1} \int_0^{T_1} R_1(t) \cos(\omega_1 t) dt = -\frac{1}{2} A(1 - \varepsilon^2 + 2q^2 A^2), \\
\tilde{R}_2 &= \frac{4}{T_2} \int_0^{T_2} R_2(t) \cos(\omega_2 t) dt \\
&= -\frac{1}{2} A(\omega^2 - \varepsilon^2 + 2q^2 A^2 \omega^2).
\end{align*}

(25)

(26)

The original amplitude–frequency formulation reads:

\[
\omega^2 = \frac{\omega_1^2 \tilde{R}_2 - \omega_2^2 \tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1}.
\]

(27)

Finally substituting Eqs. (25) and (26) into Eq. (27), the approximate frequency obtained as:

\[
\omega_{\text{AFF}} = \frac{\varepsilon}{\sqrt{1 + 2q^2 A^2}}.
\]

(28)

2.4. The differential transform method (DTM)

The basic operations of the differential transform method tabulated in Table 1.

Applying the differential transform method to the Eq. (1), the following recurrence relation is obtained:

\[
(k + 2)(k + 1)U(k + 2)
+ 4q^2 \left[ \sum_{i=0}^{k} \sum_{m=0}^{k-i} (m + 2)(m + 1)U(m + 2)U(s)U(k - s - m) \right]
+ \varepsilon^2 U(k)
+ 4q^2 \left[ \sum_{i=0}^{k} \sum_{m=0}^{k-i} (m + 1)U(m + 1)(s + 1)U(s + 1)U(k - s - m) \right] = 0,
\]

(29)

Also initial conditions in Eq. (1) transformed as:

\[
U(0) = A, U(1) = 0.
\]

(30)

From Eq. (29), for different values of \( k \), the following recursive relation is obtained:

\[
k = 0 : 2U(2) + 4q^2 [2U(0)^2 U(2)] + \varepsilon^2 U(0) + 4q^2 [U(0)[U(1)^2]] = 0,
\]

(31)

\[
k = 1 : 6U(3) + 4q^2 [U(0)[U(1)U(2) + 6U(0)^2 U(3)] + \varepsilon^2 U(1) + 4q^2 [U(0)U(1)U(2) + U(1)^3] = 0,
\]

(32)

\[
\ldots
\]

We will have:

\[
U(2) = -\frac{1}{2} \varepsilon^2 A, U(3), U(4), U(5), U(6), U(7), U(8), U(9), U(10)
\]

(33)

(34)

(35)

(36)

(37)

(38)

(39)

(40)

(41)

The above process to determine coefficients of power series is continuous and closed form solution finally obtained as:

\[
u(t) = U(0) + U(1)r + U(2)r^2 + U(3)r^3 + U(4)r^4 + \cdots
\]

(42)

3. Results and discussion

In this section, the results of the mentioned methods are compared with exact and numerical solutions. Regarding to the past sections, the EBM, HPM and AFF yield a similar form of solution and frequency-amplitude relationship. From Eqs. 8, 20 and 28 the analytical period of motion obtained as:

\[
T_{\text{anal}} = \frac{2\pi}{\omega_{\text{anal}}} = \frac{2\pi}{\varepsilon} \sqrt{1 + 2q^2 A^2}.
\]

(43)

The exact period of Eq. (1) is calculated by Wu et al. (2003):

\[
T_{\text{exact}} = \frac{4}{\varepsilon} \int_0^\pi \sqrt{1 + 4q^2 A^2 \cos^2 t} dt.
\]

(44)

Previously, He (2006) and Marinca and Herisanu (2006) obtained a similar period using max–min approach and modified iteration perturbation method, respectively. The maximum relative error of the period reach 10% even when \( qA \to \infty \). Also Marinca and Herisanu (2010) have determined a periodic solution for this Equation by means of optimal homotopy asymptotic method. The comparison between the analytic and semi-analytic methods in conjunction with the fourth-order Runge-Kutta method, presented in Figs. 1–4.

It can be clearly observed that for larger value of parameters the results of EBM, HPM and AFF show some discrepancies in comparison with the obtained results using the fourth-
order Runge–Kutta numerical method. Whereas the differential transform method can predict the solution with high accuracy even for large value of parameters. From Eq. (43), i.e., the obtained period from the governing equation of motion by the analytical approaches, we can investigate the effects of different parameters on the period. Easily be seen that when value of the $e$ decreases or value of the $q$ increases and other parameters remain constant; the period increases. Also for constant value of the $e$ when multiple values of $q,A$ are constant, the period remains without changing. One can conclude that the $e$ have opposite effect on the period of motion in comparison with $A$, $q$. To verify this issue, we consider several values of the parameter and show results in Figs. 5–7. It should be noted that these results obtained using the DTM. Moreover, the influence of constant parameters on stability and phaseplane are investigated in Figs. 8 and 9.
4. Conclusion

In this paper, nonlinear dynamics of a particle on a rotating parabola are investigated. The analytical approaches are applied via the energy balance method, homotopy perturbation method and amplitude-frequency formulation, also the semi-analytical approach implemented by the differential transform method. Results show the analytic approaches cannot predict dynamics of the particle as well as the semi-analytic approach; in contrast, the analytic methods are able to produce an explicit expression as the solution by a simple calculation, whereas each parameter is effected in the governing equation clearly, and its role can be investigated easily.

References


